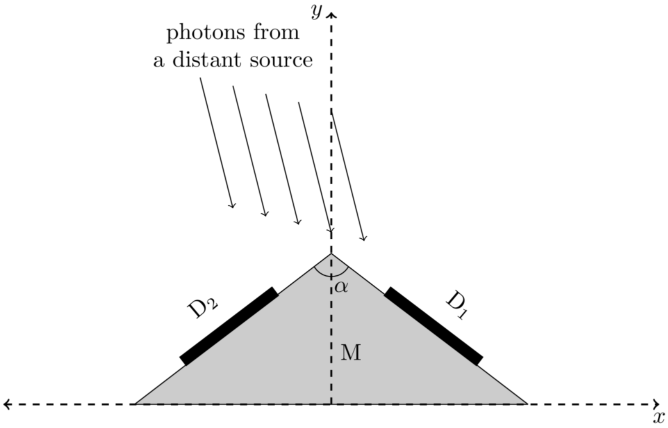
(T01) Daksha Mission **[10 marks]**

“Daksha” is a proposed Indian mission consisting of two satellites S_1 and S_2 orbiting the Earth in the same circular orbit of radius $r = 7000$ km but with 180° phase difference. These satellites observe the universe in the high-energy domain (X-rays and γ-rays). Each of the satellites of Daksha uses several flat, rectangular detectors.

To understand how to localise a source in the sky, we shall use a simplified model of the Daksha mission. Assume that S_1 has only two identical detectors D_1 and D_2, each of area $A = 0.50$ m^2. These detectors are attached to an opaque mount M as shown in the figure below. The detectors lie symmetrically around the y-axis in planes perpendicular to the x-y plane and make an angle $\alpha = 120°$ with each other.



(T01.1) When observing a distant source located in the x-y plane, detector D_1 records a power $P_1 = 2.70 \times 10^{-10}$ J s^{-1} and detector D_2 records a power $P_2 = 4.70 \times 10^{-10}$ J s^{-1}.

Estimate the angle η made by the position vector of the source with the positive y-axis, with counter-clockwise angles from the positive y-axis considered positive. **[5]**

Consider a single pulse from a distant source (not necessarily in the x-y plane) recorded by both satellites (S_1 and S_2) of Daksha. The times of the peaks of the pulses recorded by S_1 and S_2 are t_1 and t_2, respectively.

(T01.2) If $t_1 - t_2$ was measured to be 10.0 ± 0.1 ms then determine the fraction, f, of the celestial sphere where the source might lie. **[5]**

(T02) Makar-Sankranti **[10 marks]**

The festival of "Makar-Sankranti" is celebrated in India when the Sun appears to enter the zodiacal region of Capricorn (Makar = Capricorn, Sankranti = Entry) as seen from the Earth. It is currently celebrated around 14 January every year. Many years ago this festival coincided with the Winter Solstice in the northern hemisphere which we assume to take place on 21 December.

(T02.1) Based on the information above, find the year, y_c, when the celebration of this festival last coincided with the Winter Solstice in the Northern hemisphere. **[3]**

(T02.2) Given that the Sun appeared to enter the zodiacal region of Capricorn at a local time of 11:50:13 hrs on 14 January 2006 in Mumbai, calculate the date, D_{enter}, and local time, t_{enter}, of its entry into Capricorn in the year 2013. **[3]**

(T02.3) Makar-Sankranti festival is celebrated on the day of the first sunset in the zodiacal region of Capricorn. You may assume that the local sunset time for Mumbai in January is 18:30:00 hrs.

Indicate the date of celebration of the festival on every year between 2006 and 2013 (by ticking (✓) the respective box in the table given in the Summary Answersheet). **[4]**

(T03) Gravitational Waves **[15 marks]**

Orbiting binary black holes generate gravitational waves. Consider two black holes in our Galaxy, with masses $M = 36\ \mathrm{M}_\odot$ and $m = 29\ \mathrm{M}_\odot$. They revolve in circular orbits around their centre of mass with orbital angular frequency ω.

(T03.1) Assuming Newtonian gravity, derive an expression for the angular frequency, ω_{ini}, of the black hole orbits at the time t_{ini}. At this time, the separation between the black holes was 4.0 times the sum of their Schwarzschild radii. Give your answer in terms of M, m, and physical constants only.

Calculate the value of ω_{ini} (in rad s^{-1}). **[5]**

(T03.2) In general relativity, black holes in orbit emit gravitational waves with frequency f_{GW}, such that $2\pi f_{\text{GW}} = \omega_{\text{GW}} = 2\omega$. This shrinks the black hole orbits, which in turn increases f_{GW}. The rate of change of f_{GW} is

$$\frac{df_{\text{GW}}}{dt} = \frac{96\pi^{8/3}}{5} G^{5/3} c^{\beta} {M_{\text{chirp}}}^{\alpha/3} f_{\text{GW}}^{\delta/3},$$

where $M_{\text{chirp}} = \dfrac{(mM)^{3/5}}{(m+M)^{1/5}}$ is called the "chirp mass".

Find the values of α, β and δ. **[4]**

(T03.3) Assume that the gravitational waves associated with the event were first detected at time $t_{\text{ini}} = 0$. **[6]**

Derive an expression for the observed time of black hole merger, t_{merge}, when f_{GW} becomes very large. Give your answer in terms of ω_{ini}, M_{chirp}, and physical constants only.
Calculate the value of t_{merge} (in seconds).

(T04) Balmer Decrement **[15 marks]**

Consider a main-sequence star surrounded by a nebula. The observed V-band magnitude of the star is 11.315 mag. The ionised region of the nebula close to the star emits $\mathrm{H}\alpha$ and $\mathrm{H}\beta$ lines; their wavelengths are 0.6563 μm and 0.4861 μm, respectively. The theoretically predicted ratio of fluxes in $\mathrm{H}\alpha$ to $\mathrm{H}\beta$ lines is $f_{\mathrm{H}\alpha}/f_{\mathrm{H}\beta} = 2.86$. However, after this radiation passes through the outer portion of the cold dusty nebula, the observed emission fluxes of $\mathrm{H}\alpha$ and $\mathrm{H}\beta$ lines are $6.80 \times 10^{-15}\ \mathrm{W\,m^{-2}}$ and $1.06 \times 10^{-15}\ \mathrm{W\,m^{-2}}$, respectively.

The extinction A_λ is a function of wavelength and is expressed as

$$A_\lambda = \kappa(\lambda) E(B-V).$$

Here, $\kappa(\lambda)$ is the extinction curve and $E(B-V)$ denotes the colour excess in the filter bands B and V. The extinction curve (with λ in μm) is given as follows.

$$\kappa(\lambda) = \begin{cases} 2.659 \times \left(-1.857 + \frac{1.040}{\lambda}\right) + R_V, & 0.63 \le \lambda \le 2.20 \\ 2.659 \times \left(-2.156 + \frac{1.509}{\lambda} - \frac{0.198}{\lambda^2} + \frac{0.011}{\lambda^3}\right) + R_V, & 0.12 \le \lambda < 0.63 \end{cases}$$

where, $R_V = A_V/E(B-V) = 3.1$ is the ratio of total-to-selective extinction.

(T04.1) Find the values of $\kappa(\mathrm{H}\alpha)$ and $\kappa(\mathrm{H}\beta)$. **[3]**

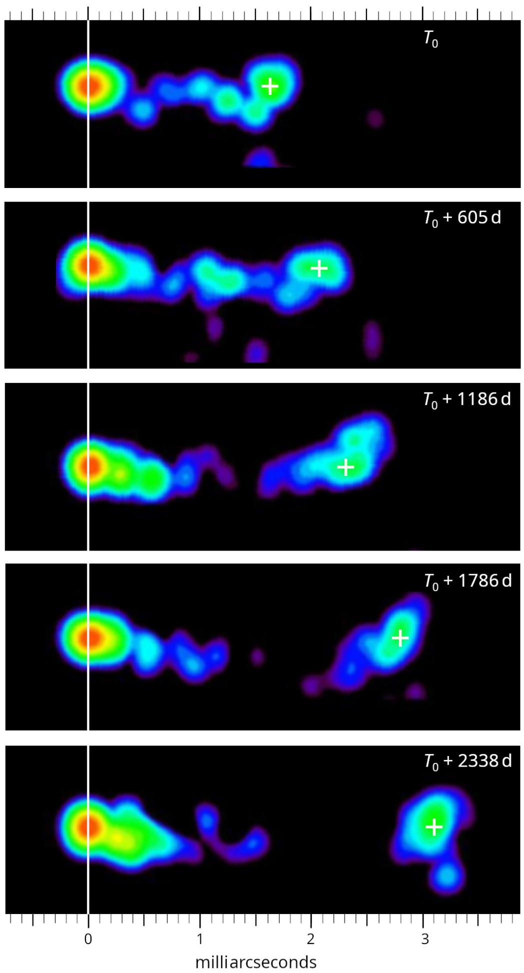
(T04.2) Find the value of the ratio of colour excess $\dfrac{E(\mathrm{H}\beta - \mathrm{H}\alpha)}{E(B-V)}$. **[4]**

(T04.3) Estimate the extinction due to the nebula, $A_{\mathrm{H}\alpha}$ and $A_{\mathrm{H}\beta}$, at Hα and Hβ wavelengths, respectively. **[6]**

(T04.4) Estimate the extinction of the nebula (A_V) and the apparent magnitude of the star in the V band, m_{V0}, in the absence of the nebula. **[2]**

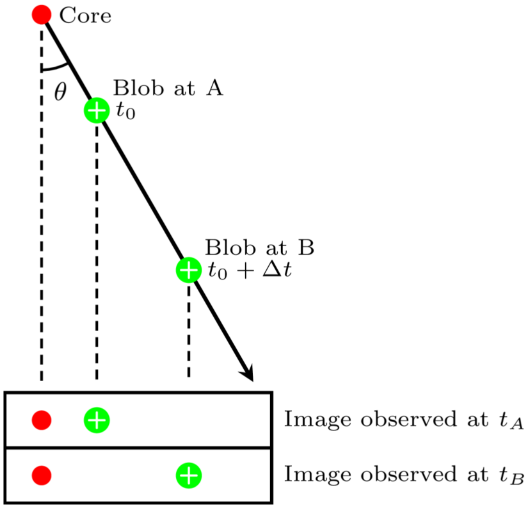
(T05) Quasars **[20 marks]**

A quasar is an extremely luminous active galaxy powered by a supermassive black hole that emits relativistic jets. The figure shows a series of panels of radio images of a quasar (with redshift $z = 0.53$, and luminosity distance $D_{\mathrm{L}} = 1.00 \times 10^{10}$ ly) at different times. The "core" aligns with the vertical white line, while a jet, consisting of a "blob" (marked white $+$), moves away from it over time. Each panel shows the observation time starting with T_0 for the first image. The angular scale is indicated at the top and bottom of the figure.



(T05.1) For each observation, determine the blob's angular separation, ϕ_{blob} (in milliarcseconds), and its transverse distance, l_{blob} (in light-years), from the quasar core. Then, calculate the blob's apparent velocity in the transverse direction (v_{app}) as a fraction of light speed, β_{app} $(= v_{\text{app}}/c)$ by using consecutive observations. Finally, calculate the average apparent velocity $\beta_{\text{app}}^{\text{ave}}$ over the entire observation period. **[5]**

The quasar jet actually moves at a relativistic speed $v \equiv \beta c$, but not necessarily in the plane of the sky; e.g., it makes an angle θ (the "viewing angle") with respect to the line of sight of a distant observer (indicated by the dashed lines), as shown in the sketch below.
For this and all subsequent parts, ignore redshift of the quasar and any relativistic effects.



(T05.2) The light emitted by the blob at two different times t_0 (corresponding to position A) and $t_0 + \Delta t$ (corresponding to position B) reaches the observer at t_{A} and t_{B}, respectively. This means that the observed (apparent) time difference is $\Delta t_{\text{app}} = t_{\text{B}} - t_{\text{A}}$.

(T05.2a) Find an expression for the ratio $\dfrac{\Delta t_{\text{app}}}{\Delta t}$ in terms of β and θ. **[2]**

(T05.2b) Using this ratio, express β_{app} in terms of β and θ. **[2]**

(T05.3) Motion is called superluminal if the apparent speed exceeds that of light ($\beta_{\text{app}} > 1$), and subluminal if it does not ($\beta_{\text{app}} < 1$).

(T05.3a) For $\beta_{\text{app}} = 1$, plot a smooth curve of β as a function of θ to mark the boundary between subluminal and superluminal motions. Shade the superluminal region in the graph with slanted lines (///). **[4]**

(T05.3b) Find the lowest true jet speed ($\beta_{\text{low}} = v_{\text{low}}/c$) for the superluminal motion to occur and also its corresponding viewing angle θ_{low}. **[2]**

(T05.4) Find an expression for the maximum viewing angle, θ_{max}, for which a given value of β_{app} will be possible. **[2]**

The core of a quasar, its central compact object, exhibits variability in its emission due to the internal processes occurring within a causally connected region. The size (radius) of this region is typically taken to be about five times the Schwarzschild radius of the core.

(T05.5) The core of a certain quasar is found to vary on time scales of about 1 h. Obtain an upper limit, $M_{\text{c, max}}$, on the mass of the central compact object, in units of solar mass. **[3]**

(T06) Galactic Rotation **[20 marks]**

The rotation curve of our Galaxy is determined using line-of-sight velocity measurements of neutral hydrogen (HI) clouds along various galactic longitudes, observed through the 21 cm HI line. Consider an HI cloud with galactic longitude l, located at a distance R from the Galactic Centre (GC) and a distance D from the Sun. Consider the Sun to be at a distance $R_0 = 8.5$ kpc from the GC. Assume that both the Sun and the HI cloud are in circular orbits around the GC in the galactic plane, with angular velocities Ω_0 and Ω, and rotational velocities V_0 and V, respectively.

The line-of-sight velocity (V_r) and transverse velocity (V_t) components of the cloud, as observed from the Sun, can be expressed as

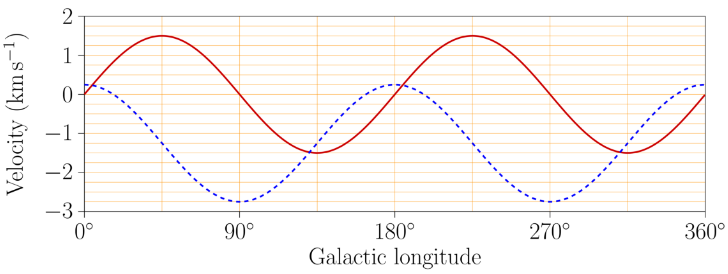
$$V_\mathrm{r} = (\Omega - \Omega_0)R_0 \sin l$$

$$V_\mathrm{t} = (\Omega - \Omega_0)R_0 \cos l - \Omega D$$

Seen from the North Galactic Pole, the Galactic rotation is clockwise. Throughout this problem, we shall take the line-of-sight velocity to be positive when receding, and clouds will be treated as point objects.

(T06.1) In the graph provided on the Summary Answersheet, sketch V_r as a function of D from $D = 0$ to $D = 2R_0$ for two lines of sight defined by (i) $l = 45°$ and (ii) $l = 135°$. Label each of your lines/curves with the value of l. **[5]**

(T06.2) The graph below shows the average radial (solid, red curve) and transverse (dashed, blue curve) velocity components of stars at a distance of 100 pc from the Sun, plotted as a function of Galactic longitude.



Using the graph, estimate the Sun's orbital period (P) around the GC in mega-years (Myr). **[3]**

(T06.3) Jan Oort noted that in the solar neighbourhood ($D \ll R_0$), the difference in the angular velocities ($\Omega - \Omega_0$) will be small, and hence, derived the following first order approximation for the line-of-sight and the transverse velocity components:

$$V_\mathrm{r} = AD \sin 2l$$
$$V_\mathrm{t} = AD \cos 2l + BD$$

where A and B are known as Oort's constants.

Let us consider two cases:
(I) the actual observed rotation curve of the Galaxy, and
(II) the rotation curve is for a hypothetical scenario where the Galaxy is devoid of dark matter, and the whole mass of the Galaxy is assumed to be concentrated at its centre.

(T06.3a) Derive expressions for the radial gradient of the rotational velocity at the location of the Sun, $\left.\dfrac{dV}{dR}\right|_{R=R_0}$, for the two cases. **[2]**

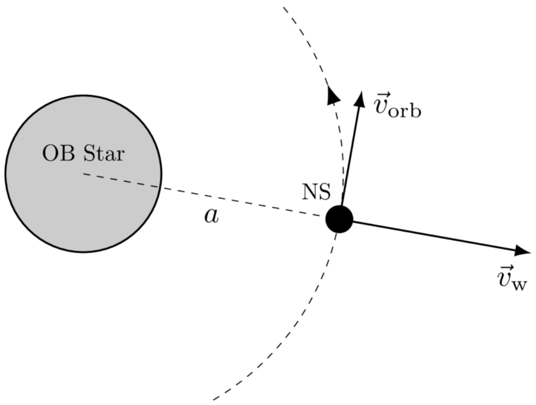
(T06.3b) Express A and B in terms of V_0, R_0, and the radial gradient of rotational velocity at the location of the Sun, $\left.\frac{dV}{dR}\right|_{R=R_0}$. **[8]**

(T06.3c) The ratio (A/B) of Oort's constants for the two given cases, (I) and (II), are defined as F_{I} and F_{II}, respectively. Determine F_{I} and F_{II}. **[2]**

(T07) Neutron Star Binary **[20 marks]**

In a binary system involving a compact star, where the companion star does not overflow its Roche lobe, the primary source of accretion for the compact star is the stellar wind from the companion star. This wind-fed accretion is especially significant in systems that include an early-type star (such as an O or B star, indicated henceforth as an OB star), alongside a compact object like a neutron star (NS) in a close orbit.

Consider such a NS-OB star binary system where a neutron star of mass $M_{\text{NS}} = 2.0\ \text{M}_\odot$ and radius $R_{\text{NS}} = 11$ km is orbiting in a circular orbit of radius a around the centre of the OB star with velocity $v_{\text{orb}} = 1.5 \times 10^5$ m s^{-1} (see figure below). Throughout this problem, the mass loss from the OB star is assumed to be spherically symmetric, and its rate is $\dot{M}_{\text{OB}} = 1.0 \times 10^{-4}\ \text{M}_\odot \text{yr}^{-1}$.



(T07.1) The accretion radius, R_{acc}, is defined as the maximum distance from the NS at which the stellar wind can be captured by the NS. If the stellar wind speed at the orbital distance of the NS is $v_{\text{w}} = 3.0 \times 10^6$ m s^{-1}, find R_{acc} for the above system in km using a standard escape velocity calculation. **[3]**

(T07.2) Assuming that all captured material is accreted by the NS, estimate the mass accretion rate, $\dot{M}_{\text{acc}}$, from the stellar wind onto the NS in units of $\text{M}_\odot \text{yr}^{-1}$ if $a = 0.5$ au. Neglect the effects of radiation pressure and the finite cooling time of the accreting gas. **[3]**

(T07.3) Now consider the situation where the stellar wind speed at the orbital distance a (near the NS) becomes comparable with orbital speed of the NS. The mass accretion rate from the stellar wind onto the NS in this case would be given by an expression of the form $\dot{M}_{\text{acc}} = \dot{M}_{\text{OB}} f(\tan\beta, q)$, where $q = M_{\text{NS}}/M_{\text{OB}}$ is the mass ratio of the binary and β is the angle between the wind velocity direction and radial direction away from the OB star in the frame of the NS. Obtain the expression for $f(\tan\beta, q)$ assuming $M_{\text{OB}} \gg M_{\text{NS}}$. **[6]**

(T07.4) Consider that the fully ionised material accretes radially and is hindered due to the strong magnetic field $\vec{B}$ of the NS. This effect can be modelled as a pressure, given by $\frac{B^2}{2\mu_0}$. We shall assume that the NS has a dipole magnetic field whose magnitude in the equatorial plane varies with the distance r from the NS for $r \gg R_{\rm NS}$ as

$$B(r) = B_0 \left(\frac{R_{\rm NS}}{r} \right)^3$$

where B_0 is the magnetic field at the equator of the NS. Assume that the axis of the magnetic dipole aligns with the rotation axis of the NS.

(T07.4a) Obtain the magnetic pressure, $P_{\rm eq,\,mag}$, in the equatorial plane in terms of B_0, $R_{\rm NS}$, r, and other suitable constants. **[1]**

(T07.4b) The maximum distance where the accretion flow is stopped by the magnetic field at the equatorial plane is called the magnetospheric radius $R_{\rm m}$. This flow of matter will exert a pressure due to the relative motion between the incoming stellar wind and the NS. Obtain an approximate expression for the critical magnetic field $B_{0,\,\rm c}$ for which $R_{\rm m}$ coincides with $R_{\rm acc}$ and calculate its value in Tesla. Magnetic effects are neglected for $r > R_{\rm m}$ and consider $v_{\rm w} \gg v_{\rm orb}$. **[7]**

(T08) Shadow of a black hole **[20 marks]**

The Event Horizon Telescope (EHT) has released an image of the supermassive black hole at the centre of the M87 galaxy, as shown in the left panel of the figure below.

To understand some simple features of this image, we will consider a simplified model of a non-rotating, static, spherically symmetric, black hole of mass $M = 6.5 \times 10^9 \mathrm{M}_\odot$, surrounded by a massless, thin, planar accretion disk of inner and outer radii, $a_{\rm inner} = 6R_{\rm SC}$ and $a_{\rm outer} = 10R_{\rm SC}$, respectively, where $R_{\rm SC}$ is the Schwarzschild radius. A face-on view sketch is shown in the right panel of the figure below (figure is not to scale).

We assume that the accretion disk is the only source of light to be considered. Every point on the disk emits light in all directions. This light travels under the influence of the gravitational field of the black hole. The path of the light rays is governed by two equations given below (which are similar to those of an object around the Sun):

$$\frac{1}{2}v_{\rm r}^2 + \frac{L^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) = E \quad ; \qquad v_\phi = r\,\omega = \frac{L}{r}$$

where $r \in (R_{\rm SC}, \infty)$ is the radial coordinate, $\phi \in [0, 2\pi)$ is the azimuthal angle, and E and L are constants related to the conserved energy and conserved angular momentum, respectively.

Here $v_{\rm r} \equiv dr/dt$ is the magnitude of the radial velocity, v_ϕ is the magnitude of the tangential velocity, and $\omega \equiv d\phi/dt$ is the angular velocity. We define the impact parameter b for a trajectory as $b = L/\sqrt{2E}$. Time dilation is neglected in this problem.

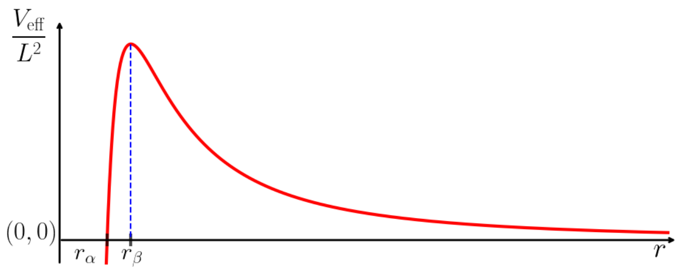
Another useful equation is obtained by differentiating the first equation:

$$\frac{dv_\mathrm{r}}{dt} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2r^4} = 0$$

(T08.1) Circular light trajectories can exist around the black hole. Find the radius, r_ph, and impact parameter, b_ph, for such photon trajectories in terms of M and relevant constants. **[4]**

(T08.2) Calculate the time taken, T_ph, for a photon to complete one full orbit in the circular light trajectory in seconds. **[2]**

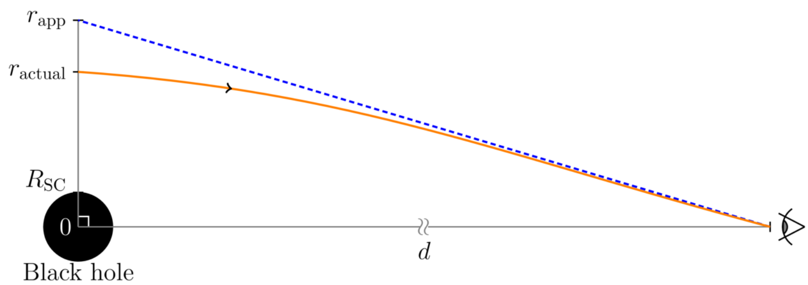
(T08.3) The radial velocity equation given above (the first equation in this question) can be compared with an equation of the form $\frac{v_\mathrm{r}^2}{2} + V_\mathrm{eff}(r) = E$ for light trajectories. A schematic plot of V_eff/L^2 as a function of r is given below.



(T08.3a) The plot indicates two special radii, r_α and r_β. Obtain expressions for r_α and r_β in terms of M and relevant constants. **[2]**

(T08.3b) A photon travelling inward from the accretion disk towards the black hole can still escape out to infinity in some cases. Find the expression for the smallest value of the turning point radius, r_t, for such a photon, in terms of M and relevant constants. Find the expression for the minimum value of the impact parameter, b_min, for this photon. **[3]**

(T08.4) A ray of light coming from a radius r_actual from the centre of the system in the plane of the sky will suffer strong bending due to the gravity of the black hole, and eventually reach an observer (denoted by an eye) at a large distance d from the system, as shown below.

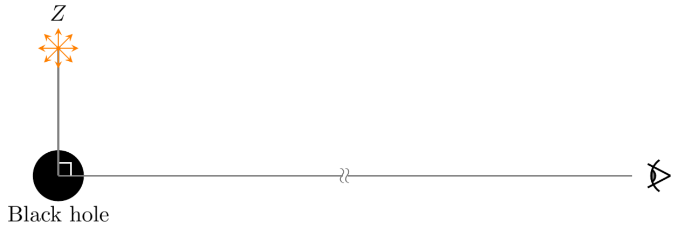


To this observer, the ray would appear to have originated from a different point at a distance $r_\mathrm{app} \approx b$ from the black hole's centre in the plane of the sky, where b is the impact parameter for that photon trajectory. For points on the accretion disk at $r = r_\mathrm{actual}$, one may assume the following relation:

$$b(r_\mathrm{actual}) \approx r_\mathrm{actual}(1 + R_\mathrm{SC}/r_\mathrm{actual})^{1/2}$$

For the distant observer, like ourselves, with a face-on view of the accretion disk, the image of the system will appear to be circularly symmetric in the plane of the sky. Determine the outermost apparent radius, r_outer, and the innermost apparent radius, r_inner, of the image in units of au. **[5]**

(T08.5) Consider an isolated supermassive black hole of mass $M = 6.5 \times 10^9$ $M_\odot$ without any accretion disk. A brief strong burst of electromagnetic radiation occurs for 5 s at a point Z at a distance, say, $r_Z = 6R_{SC}$ from the black hole as shown in the figure. The burst at point Z emits light in all directions. An observer at a point far from the black hole (denoted by an eye in the figure below) takes a long exposure image of the region around the black hole for 60 s.



Choose the correct option for each of the statements below:

(T08.5a) The number of possible paths for light to travel from Z to the observer is **[2]**
(A) At most one (B) Exactly one (C) Exactly two (D) Greater than two.

(T08.5b) The number of images of the EM burst at Z that will be seen in the long exposure image is **[2]**
(A) At most one (B) Exactly one (C) Exactly two (D) Greater than two.

(T09) Atmospheric Seeing **[35 marks]**

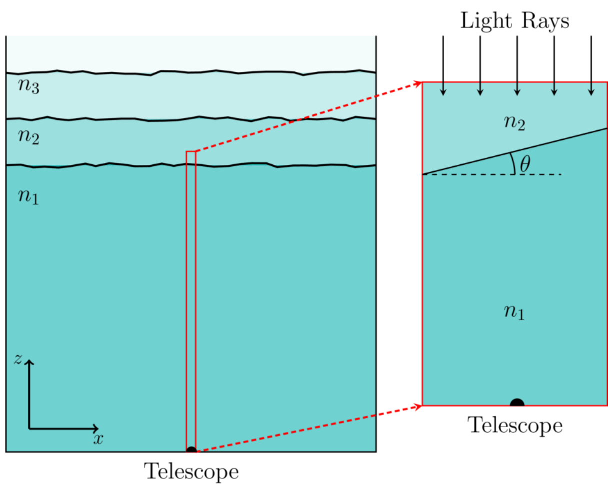
A telescope with an achromatic convex objective lens of diameter $D = 15$ cm and focal length $f = 200$ cm is pointed to a star at the zenith.

(T09.1) Find the diameter (in m), d_{image}, of the image of a point source as produced by the objective lens at its focal plane for green light ($\lambda = 550$ nm), considering only the effects of diffraction. **[1]**

The image of an astronomical source is also affected by the so-called "atmospheric seeing".

The boundaries between the layers in the atmosphere, as well as the refractive indices of the layers, change continuously due to turbulence, temperature variation and other factors. This leads to tiny changes in the position of the image in the focal plane of the telescope, known as the "twinkling effect". For the rest of the problem, apart from using the finite diffraction-limited size of the image of the star (as used above), no interference effects will be considered.

The left panel of the figure below shows a vertical cross-section of the atmosphere with multiple layers of different refractive indices ($n_1, n_2, n_3, \ldots$). The right panel shows the zoomed in view of a thin vertical segment of the atmosphere and the boundary between the two lowest atmospheric layers of refractive indices n_1 and n_2 ($n_1 > n_2$). We consider only these two layers and their boundary for this problem. The diagrams are not to scale.



(T09.2) Let the boundary between the two layers be at a height $H = 1$ km directly above the telescope objective, with a tilt of $\theta = 30°$ with respect to the horizontal plane. In all parts of this problem θ is taken to be positive in the anti-clockwise direction. For a monochromatic light source, $n_1 = 1.00027$ and $n_2 = 1.00026$. Let the angular shift of the image at the focal plane of the telescope for a star at the zenith be α.

(T09.2a) Draw an appropriately labelled ray-diagram at the boundary showing n_1, n_2, θ and α. **[2]**

(T09.2b) Find the expression for α in terms of θ, n_1 and n_2. Use the small angle approximations: $\sin\alpha \approx \alpha$ and $\cos\alpha \approx 1$. **[2]**

(T09.2c) Calculate the displacement, Δx_θ (in m), in the position of the image if θ increases by 1% (keeping n_1 and n_2 fixed). **[3]**

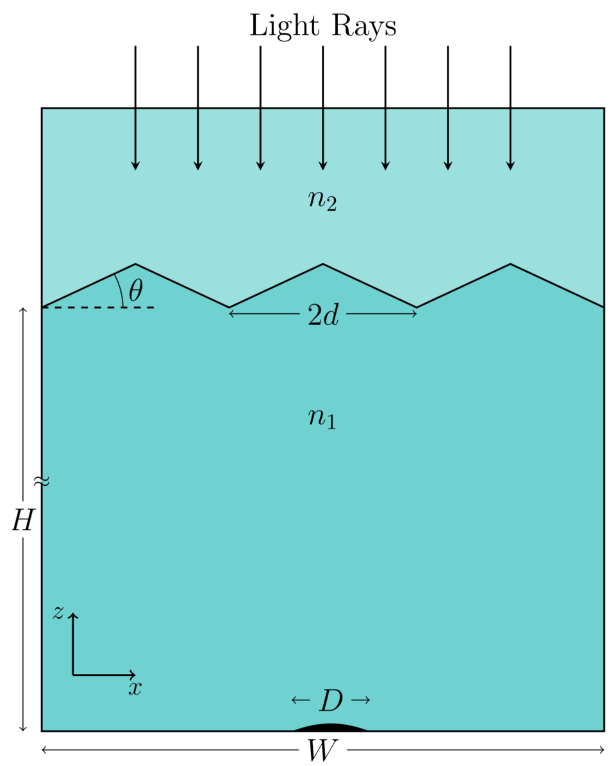
(T09.2d) Calculate the displacement, Δx_n (in m), in the position of the image if n_2 increases by 0.0001% (keeping n_1 and θ fixed). **[3]**

(T09.3) For white light coming from a star at the zenith, choose which of the following rows most closely describes the shape and colour of the image by ticking (✓) the appropriate box (only one) in the Summary Answersheet. Note x increases from left to right in the figure. **[2]**

	Image colour	Image shape	Left edge	Right edge
A	White	Circular		
B	White	Elliptical		
C	Coloured	Circular	Blue	Red
D	Coloured	Circular	Red	Blue
E	Coloured	Elliptical	Blue	Red
F	Coloured	Elliptical	Red	Blue

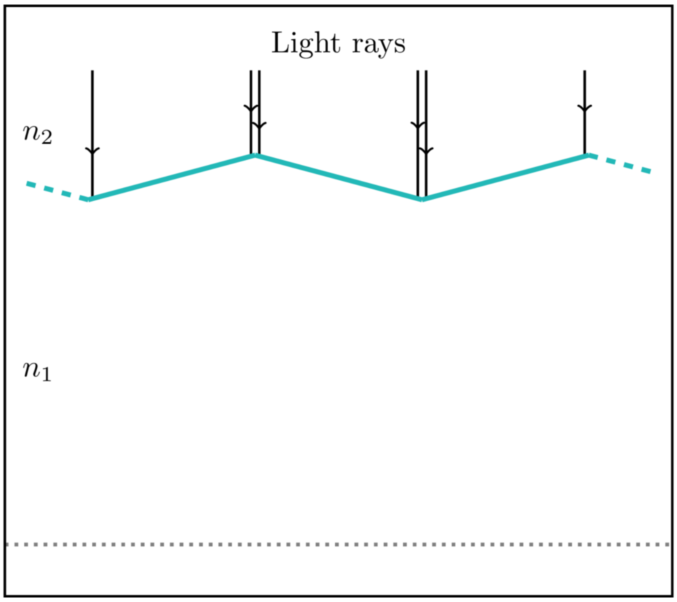
For all remaining parts of this question, we consider monochromatic green light with $\lambda = 550$ nm. We model the boundary between the layers as a set of infinite zigzag planes (running perpendicular to the plane of the page) separated by $d = 10$ cm along x-axis, with either $\theta = 10°$ or $\theta = -10°$.

The figure below (not to scale) shows a cross-section of this model of the atmosphere of width W ($W \ll H$). For telescopes with large apertures, this zigzag nature of the boundary results in the formation of speckles in the focal plane.



(T09.4) Consider an atmosphere modelled as above.

(T09.4a) A section of the atmosphere with consecutive zigzag planes, with the same parameters as stated above, is shown in the diagram below (not to scale).



In the copy of this diagram on the Summary Answersheet, draw the paths of the incident light rays down to the plane where the telescope objective is placed, shown by the grey dotted line on the base of the diagram.

Mark the region(s), if any, by "X" in the diagram where no light rays will reach. **[4]**

(T09.4b) Calculate the width W_X of such region(s). **[3]**

(T09.4c) By appropriately choosing the location of the telescope relative to the structure of the boundary, find the largest diameter, D_{max}, of the telescope objective with which it will be possible to obtain a single image of a star, **[4]**

(T09.5) Consider the case when the zigzag shape of the boundary is allowed in both x and y directions (like a field of pyramids), and $D = 100$ cm (with $f = 200$ cm). **[6]**

Draw the qualitative pattern of the resulting speckles in the box given in the Summary Answersheet.

(T09.6) For a turbulent atmosphere, consider the same zigzag shape running parallel to the boundary layer only along x-direction, but now the angle between the two planes is changing from 10° to $-10°$ in 1.0 s, at a uniform rate. Assume that this results in a uniform rate of shift in the position of the image. **[5]**

Consider a telescope with $D = 8$ cm and $f = 1$ m. Estimate the longest exposure time t_{max} allowed for its CCD camera so that only a single image is obtained, and any possible deviation in the image's position remains less than 1% of the diffraction-limited diameter of the image.

(T10) Big Bang Nucleosynthesis **[35 marks]**

During the radiation-dominated era in the early Universe, the scale factor of the Universe $a \propto t^{1/2}$, where t is the time since the Big Bang. During most of this era, neutrons (n) and protons (p) remain in thermal equilibrium with each other via weak interactions. The number density (N) of free neutrons or protons is related to the temperature T and their corresponding masses m such that

$$N \propto m^{3/2} \exp\left(-\frac{mc^2}{k_B T}\right),$$

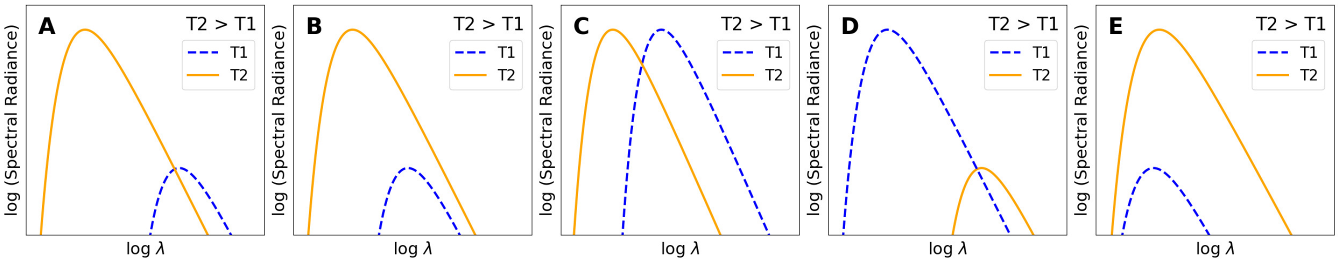
as long as time $t \leq t_{wk}$ =1.70 s, when $k_B T \geq k_B T_{wk}$ =800 keV. After t_{wk}, the weak interactions can no longer maintain such an equilibrium, and free neutrons decay to protons with a half-life time of 610.4 s.

(T10.1) Let the number density of protons be $N_{\rm p}$, and that of neutrons be $N_{\rm n}$. Calculate the relative abundance of neutrons given by the ratio $X_{\rm n,\,wk} = N_{\rm n}/(N_{\rm n} + N_{\rm p})$ at time $t_{\rm wk}$. **[4]**

(T10.2) Photons maintain thermal equilibrium and retain a blackbody spectrum at all epochs.

(T10.2a) Find the index β, such that $T(a) \propto a^{\beta}$. **[2]**

(T10.2b) Identify which of the following graphs shows the correct behaviour of the spectral energy density for two temperatures T_1 and T_2. Tick (✓) the correct option in the Summary Answersheet. **[2]**



(T10.3) After $t_{\rm wk}$, the process of formation of deuterium from protons and neutrons is governed by the Saha equation, given by the Indian physicist Prof. Meghnad Saha, which can be simplified to

$$\frac{N_{\rm D}}{N_{\rm n}} = 6.5\eta\left(\frac{k_{\rm B}T}{m_{\rm n}c^2}\right)^{3/2} \exp\left(-\frac{(m_{\rm D} - m_{\rm p} - m_{\rm n})c^2}{k_{\rm B}T}\right).$$

Here, the baryon-to-photon ratio η is 6.1×10^{-10}, and $N_{\rm D}$ is the number density of deuterium.

(T10.3a) Plot the ratio $N_{\rm D}/N_{\rm n}$ on the grid in the Summary Answersheet, for at least 4 reasonably spaced values of temperature that lie in the domain $k_{\rm B}T = [60, 70]$ keV, and draw a smooth curve passing through these points. **[5]**

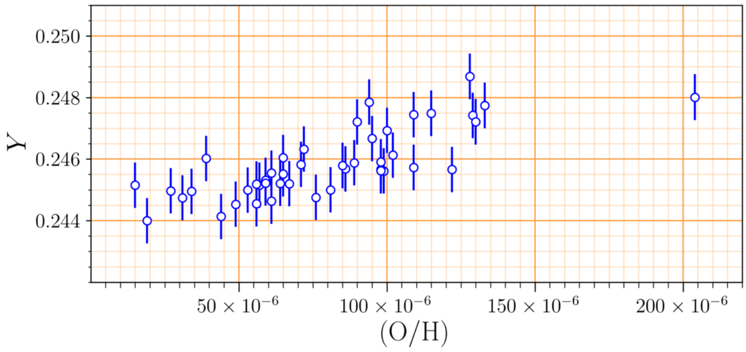
(T10.3b) From the plot find $k_{\rm B}T_{\rm nuc}$ (in keV) when $N_{\rm D} = N_{\rm n}$. **[1]**

(T10.3c) Instead, now assume that all the free neutrons combine instantaneously with the protons at $k_{\rm B}T_{\rm nuc}$ to form Deuterium, which is immediately converted to Helium ($^4_2$He). Compute the corresponding time of nucleosynthesis, $t_{\rm nuc}$ (in s), for the formation of Helium. **[4]**

(T10.4) Calculate the value of $X_{\rm n,\,nuc}$ immediately before $t_{\rm nuc}$. **[5]**

(T10.5) The primordial Helium abundance, $Y_{\rm prim}$, is defined to be the fraction of total baryonic mass in the Universe that is bound in Helium just after $t_{\rm nuc}$. Obtain a theoretical estimate for the value of $Y_{\rm prim}$. For the purpose of this calculation alone, assume $m_{\rm p} \approx m_{\rm n}$ and that the mass of Helium, $m_{\rm He} \approx 4m_{\rm n}$. **[3]**

(T10.6) The primordial abundance of Helium is very difficult to measure, as stars continuously convert Hydrogen to Helium in the Universe. The amount of processing by stars in a galaxy is characterised by the relative number density of Oxygen (which is only produced by stars) to Hydrogen, denoted as (O/H), in the galaxy. A compilation of the measurements of (O/H) and the Helium abundance, Y, for different galaxies is plotted below.



Use the points in this plot (which is reproduced in the Summary Answersheet) to answer the following.

(T10.6a) Estimate Y for a blue compact dwarf galaxy with a value of (O/H)=1.75×10^{-4}. **[2]**

(T10.6b) Obtain the slope $dY/d(\mathrm{O/H})$ of the straight line fit to the above data. **[2]**

(T10.6c) Estimate the primordial Helium abundance, $Y_{\mathrm{prim}}^{\mathrm{obs}}$, based on the above data. **[2]**

(T10.7) The deviation between Y_{prim} and $Y_{\mathrm{prim}}^{\mathrm{obs}}$ can be reconciled by changing the baryon-to-photon ratio η. When η is decreased, as indicated by $\downarrow$ in the Summary Answersheet, indicate the increase ($\uparrow$) or decrease ($\downarrow$) in $N_{\mathrm{D}}/N_{\mathrm{n}}(T)$, T_{nuc} (when $N_{\mathrm{D}} = N_{\mathrm{n}}$), t_{nuc}, $X_{\mathrm{n,\,nuc}}$, and Y_{prim} in the boxes provided in the Summary Answersheet. **[3]**

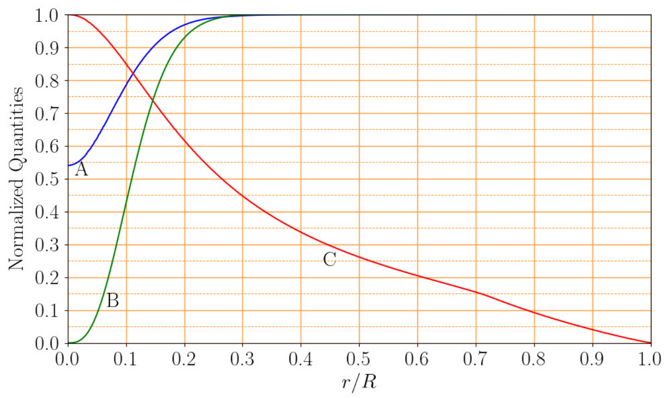
(T11) Stars through graphs **[50 marks]**

Stars can be well approximated as spherically symmetric objects, and hence the radial distance, r, from the centre can be chosen as the only independent variable in modelling stellar interiors. The mass contained within a sphere of radius, r, is denoted by $m(r)$. The luminosity, $l(r)$, is defined as the net energy flowing outward through a spherical surface of radius r per unit time. Other quantities of interest, for example, the density $\rho(r)$, temperature $T(r)$, hydrogen mass fraction $X(r)$, helium mass fraction $Y(r)$, and the nuclear energy generated per unit mass per unit time $\epsilon_{\rm nuc}(r)$, are taken to be functions of r. Throughout this problem, we shall neglect the effects of diffusion and gravitational settling of elements inside the star.

The symbol "log" refers to the logarithm to the base 10. The problem consists of three independent parts.

(T11.1) **Part 1: Inside a star**

The graph below shows the variation of three structural quantities, A, B, and C, as functions of the fractional radius r/R in a stellar model of mass 1 $M_\odot$ and age 4 GYr, where R is the photospheric radius of the star. The values of the helium mass fraction at the (photospheric) surface, Y_s, and the metallicity (mass fraction of all elements heavier than helium) at the (photospheric) surface, Z_s, of the star are given by $(Y_s, Z_s) = (0.28, 0.02)$. All quantities shown in the plots are normalised by their respective maximum values.



(T11.1a) Identify the three quantities A, B, and C uniquely from among the five possibilities: **[6]**

$$T(r),\ l(r),\ \epsilon_{\rm nuc}(r),\ X(r),\ Y(r).$$

(Write A/B/C in the boxes beside the appropriate quantities in the Summary Answersheet. No justification is needed for your answer.)

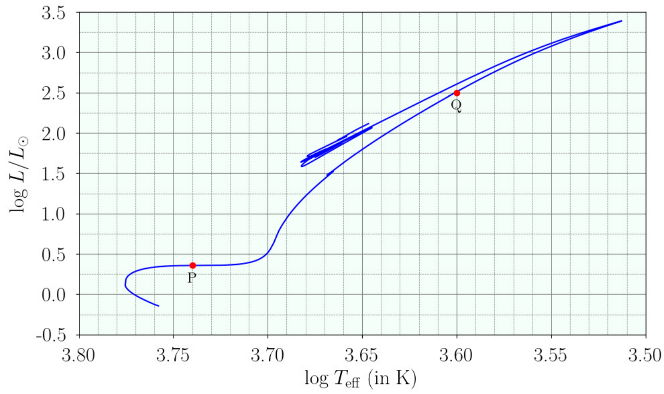
(T11.1b) What is the mass fraction of helium at the centre, Y_c, of the star? **[3]**

(T11.1c) Sketch the remaining two quantities from the list of five (which were not identified as curves A, B, or C) given in (T11.1a), as functions of r/R on the same graph in the Summary Answersheet. Label your curves to indicate their respective quantities. **[5]**

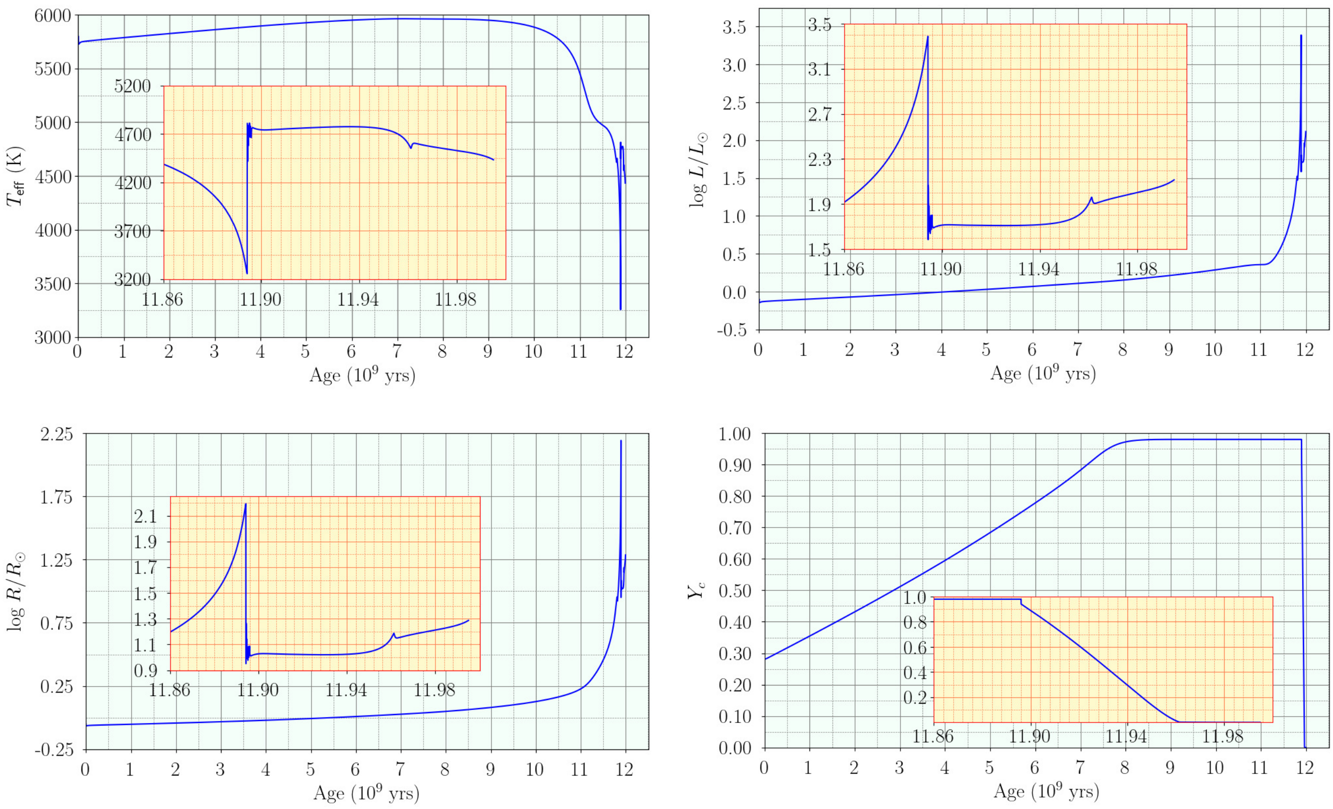
(T11.2) **Part 2: Evolving stars**

Consider the evolution of a $1 M_\odot$ star whose initial uniform composition is given by the mass fractions of helium, $Y_0 = 0.28$, and metals, $Z_0 = 0.02$. The figures below show the variation of different global quantities of this star as it evolves from ZAMS (Zero Age Main Sequence) until it finishes burning helium in its core.

The graph below shows the evolutionary track of the star on the HR diagram (plot of $\log L/L_\odot$ vs $\log T_{\rm eff}$, where L is the surface luminosity and $T_{\rm eff}$ is the effective temperature).



The figure below has four graphs which show the variation of T_{eff} (in K), L (plotted as $\log L/L_\odot$), R (plotted as $\log R/R_\odot$), and Y_c with age (in 10^9 yr) of the same star. In each of these four graphs, the insets show the variations of the respective quantities in detail between the ages of 11.86×10^9 yr to 12.00×10^9 yr, for greater clarity.



Use these graphs to answer the questions below.

(T11.2a) What is the approximate main sequence lifetime, t_{MS} (in years), of the star? **[1]**

(T11.2b) What is the approximate duration, Δt_{He} (in years), for which the star burns helium in its core? **[1]**

(T11.2c) What fraction, f_{H}, of the initial amount of hydrogen at its centre has been burnt when the luminosity of the star is 1 $L_\odot$? **[3]**

(T11.2d) What is the radius of the star, R_1, when 60% of the initial amount of hydrogen at its centre has been burnt? Give your answer in units of $R_\odot$. **[3]**

(T11.2e) What are the radii of the star, R_{P} and R_{Q}, corresponding to its positions P and Q, respectively, as marked on the HR diagram? Give your answer in units of $\mathrm{R}_{\odot}$. **[4]**

(T11.3) **Part 3: Mass distribution inside a star**

The equation that governs the distribution of mass inside a star is given by

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

It would be convenient to express this equation in terms of three dimensionless variables, namely, the fractional mass, q, the fractional radius, x, and the relative density, σ, that we define as

$$q = m/M \qquad x = r/R \qquad \sigma = \rho/\bar{\rho}$$

where M and R are the total mass and radius of the star, respectively, and $\bar{\rho} \equiv \dfrac{M}{\frac{4}{3}\pi R^3}$ is the average density of the star. For the particular star that we shall be considering in this part, the following information is given:

- The central density $\rho(x=0) = 80\bar{\rho}$
- Half of the star's mass is contained within the inner 25% of its total radius, and 70% of its mass is contained within the inner 35% of its total radius.

In all subsequent parts of this question, it will be sufficient to round off all derived numerical coefficients to within 0.005.

(T11.3a) Express the above equation describing the dependence of mass on radius in terms of x, $\dfrac{dq(x)}{dx}$ and $\sigma(x)$. **[2]**

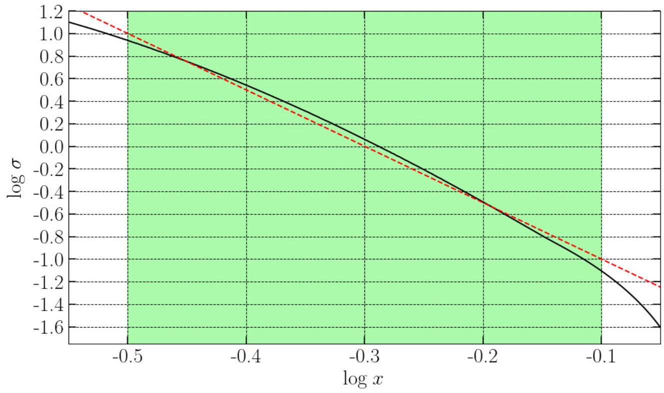
To obtain the distribution of mass with radius, we need to know the density profile inside the star. For the purpose of this problem, we shall describe the variation of density with radius by approximate forms in two domains of x:

- the inner part of the star: $0 \le x \le 0.32$
- the middle part of the star: $0.32 < x < 0.80$

We do not make any approximation for the outermost part, i.e., $0.80 \le x \le 1.00$.

(T11.3b) **Approximation for the middle part**:

The variation of $\log\sigma$, as a function of $\log x$ in the middle part of the star, is shown (by the black curve) in the graph below. We shall make a linear approximation (shown as a dashed red line in the graph) for $\log\sigma$ as a function of $\log x$ in the domain $-0.5 < \log x < -0.1$, i.e., $0.32 \lesssim x \lesssim 0.80$ (shown by the green shaded domain). We shall approximate the slope of this line by the nearest integer.



Use this approximation to write an expression for $\sigma(x)$ as a function of x in the domain $0.32 < x < 0.80$. **[4]**

(T11.3c) Use the result of (T11.3b) to derive an expression for $q(x)$ in the domain $0.32 < x < 0.80$. **[6]**

(T11.3d) **Approximation for the inner part:**

In the inner part of the star ($0 \leq x \leq 0.32$), the density may be approximated as a linear function of radius, i.e., $\sigma(x) = Ax + B$, where A, B are constants. Determine A and B, and hence obtain an expression for $q(x)$ in the domain $0 \leq x \leq 0.32$. Note that the approximations adopted in the previous part and this part may lead to small discontinuities in density or mass at $x = 0.32$. **[8]**

(T11.3e) The expressions for $q(x)$ obtained in parts (T11.3c) and (T11.3d) are approximations that describe the variation of mass with radius quite well, but only in specific regions of the star. For the domain $0.80 \leq x \leq 1$ (for which we have not derived any expression), it is possible to use appropriate extrapolation from the neighbouring region. Use these approximate expressions and given data to sketch a smooth curve (without any discontinuities either in $q(x)$ or its derivative) for $q(x)$ vs x for the entire star ($0 \leq x \leq 1$), that represents the variation of mass with radius. **[4]**

(T12) Hawking Radiation from Black Holes **[50 marks]**

(T12.1) A black hole (BH) typically forms by the gravitational collapse of a massive star at the end of its life cycle to a point called a singularity. Due to the extreme gravity of such an object, nothing that enters the so-called event horizon (a spherical surface with $r = R_{\mathrm{SC}}$, where r is the distance from the singularity) is able to escape from it. Here, R_{SC} is referred to as the Schwarzschild radius.

(T12.1a) **Modelling the origin of Hawking radiation:** Consider a pair of particles, each with mass m , produced on either side of the BH horizon. One particle is slightly outside the horizon at $r \approx R_{\mathrm{SC}}$, while the other particle is inside the horizon at $r = \kappa R_{\mathrm{SC}}$. Assume that the total energy of a particle is the sum of its rest mass energy mc^2 and the gravitational potential energy due to the BH.

Determine the value of κ for which the particle pair has zero total energy. **[4]**

(T12.1b) **Temperature of a black hole:** If the particle produced outside the horizon in the above process has enough kinetic energy, it may escape the BH in a process called Hawking radiation. The one inside the horizon, which has negative energy, gets absorbed and decreases the mass of the BH.

Assume that all Hawking radiation is made of photons which form a black body spectrum that peaks at the wavelength $\lambda_{\rm bb} \approx 16 R_{\rm SC}$. It is known that for a solar mass BH, $R_{\rm SC,\odot}$ =2.952 km.

Obtain an expression for the temperature, $T_{\rm bh}$, of the BH corresponding to this black body radiation. Give your answer in terms of the mass $M_{\rm bh}$ and physical constants. Calculate the Schwarzschild radius, $R_{\rm SC,\,10\odot}$, and temperature, $T_{\rm bh,\,10\odot}$, for a BH with mass 10 $\rm M_\odot$. **[4]**

(T12.1c) **Mass loss of a black hole:** Assume that the Hawking radiation is emitted out from the event horizon.

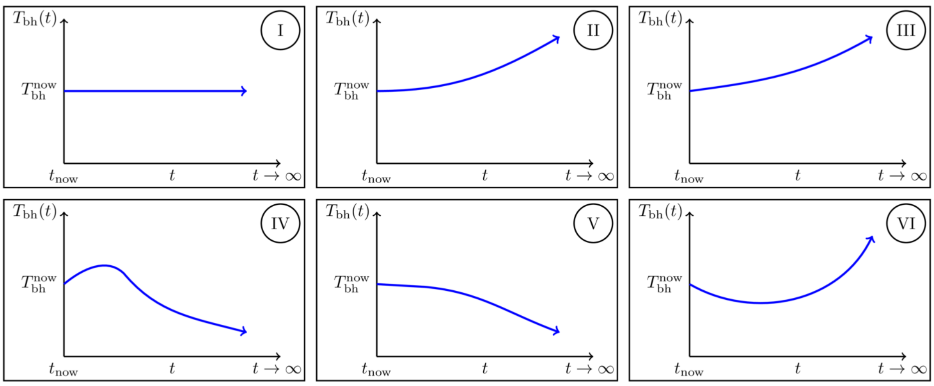
Using the mass-energy equivalence, obtain an expression for the rate of mass loss, $dM_{\rm bh}(t)/dt$, in terms of the mass $M_{\rm bh}(t)$ of the BH and physical constants.

Hence, obtain an expression for $M_{\rm bh}(t)$ for a BH with initial mass M_0. Sketch $M_{\rm bh}(t)$ as a function of t from $M_{\rm bh} = M_0$ to $M_{\rm bh} = 0$. **[8]**

(T12.1d) **Lifetime of a black hole:** Obtain an expression for the lifetime $\tau_{\rm BH}$ at which a black hole with initial mass M_0 completely evaporates due to Hawking radiation, in terms of M_0 and physical constants. Calculate the lifetime $\tau_{\rm bh,10\odot}$ (in seconds) for a black hole with $M_0 = 10\,\rm M_\odot$. **[3]**

(T12.1e) **Black hole in a CMB radiation bath:** Consider an isolated black hole in space, far away from other bodies. The black hole has current temperature $T^{\rm now}_{\rm bh}$, and is surrounded by the cosmic microwave background (CMB), which has current temperature $T^{\rm now}_{\rm cmb} = 2.7$ K. The black hole can grow in mass by absorbing CMB radiation, and lose mass due to Hawking radiation.

Taking into account the accelerating expansion of the Universe, identify which of the following figures below show the long-term time evolution of $T_{\rm bh}$ in the following three cases:
(X) $T^{\rm now}_{\rm bh} > T^{\rm now}_{\rm cmb}$, (Y) $T^{\rm now}_{\rm bh} = T^{\rm now}_{\rm cmb}$, (Z) $T^{\rm now}_{\rm bh} < T^{\rm now}_{\rm cmb}$.



Match each case, X, Y and Z to the appropriate graph. Express your answer in the Table given in the Summary Answersheet by ticking one box in each row. **[6]**

(T12.2) Primordial black holes (PBHs) of much smaller masses were able to form in the very early Universe. All the following questions are related to PBHs. Here, neglect any processes that increase the mass of the black hole.

(T12.2a) **PBH evaporating at the current epoch:** As you may have noticed from the answers to the previous questions, solar mass black holes would take a long time to evaporate. However, since PBHs can have a much smaller mass, we may be able to see them evaporating in current times.

Find the initial mass $M_{0,\,\rm PBH}$ (in kg), Schwarzschild radius $R_{\rm SC,\,PBH}$ (in m), and temperature $T_{\rm PBH}$ (in K) of a black hole which can evaporate completely at the present epoch, i.e. one with lifetime $\tau_{\rm PBH} = 14$ billion years. **[4]**

(T12.2b) **Formation of a PBH:** In the radiation-dominated early Universe, the scale factor varies as $a(t) \sim t^{1/2}$. In this era, PBHs form due to the collapse of all energy contained in a region of physical size ct, where t is the age of the Universe at that time.

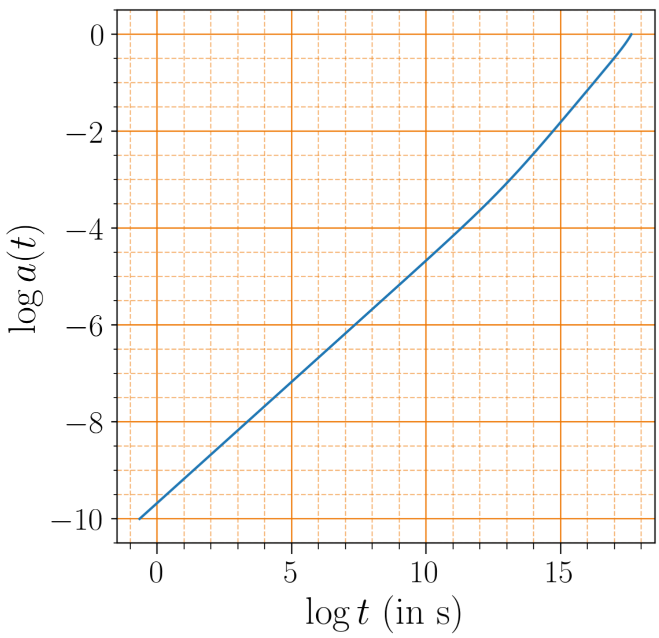
A PBH with mass of 1×10^{12} kg forms when the age of the Universe is about 1×10^{-23} s. Calculate the age of the Universe, t_{20}, when a PBH of mass 1×10^{20} kg forms. **[6]**

(T12.2c) **Observed spectrum of Hawking radiation from PBH:** Consider a PBH of initial mass 1×10^{10} kg which completely evaporates at the end of its lifetime τ_{PBH}. For this part, assume for simplicity that most of the Hawking radiation is emitted at this time, with a temperature corresponding to its initial mass. Take the scale factor of the Universe to be evolving as $a(t) \sim t^{2/3}$.

Calculate the peak wavelength, λ_{earth}, of this Hawking radiation as observed at Earth at the present epoch (at $t = 14$ billion years). **[5]**

(T12.2d) **High energy cosmic radiation from PBH:** Now assume that the Hawking radiation emitted at a given time t corresponds to photons emitted with an energy $k_{\text{B}}T_{\text{bh}}(t)$. Also, the highest possible temperature for a black hole is the Planck temperature T_{Planck} where $k_{\text{B}}T_{\text{Planck}} = 1 \times 10^{19}$ GeV.

The evolution of the scale factor over relevant time scales is given in the following figure. The scale factor today is set to be unity. $t(s)$ on the time axis represents the age of the universe in seconds.



If a photon with an energy of $E_{\text{det}} = 3.0 \times 10^{20}$ eV is observed on Earth, determine the largest and the smallest possible values of the initial mass of the PBH (M_0^{max} and M_0^{min}, respectively) which could be responsible for this photon. **[10]**