(D01) 30 Years of Exoplanets **[90 marks]**

This problem explores some aspects of the two main methods of exoplanet detection: radial velocity and transit light curves. Throughout this problem we shall consider a particular system of a single planet (P) in a circular orbit with radius a around a solar-type star (S). We shall refer to this system as the "SP system".

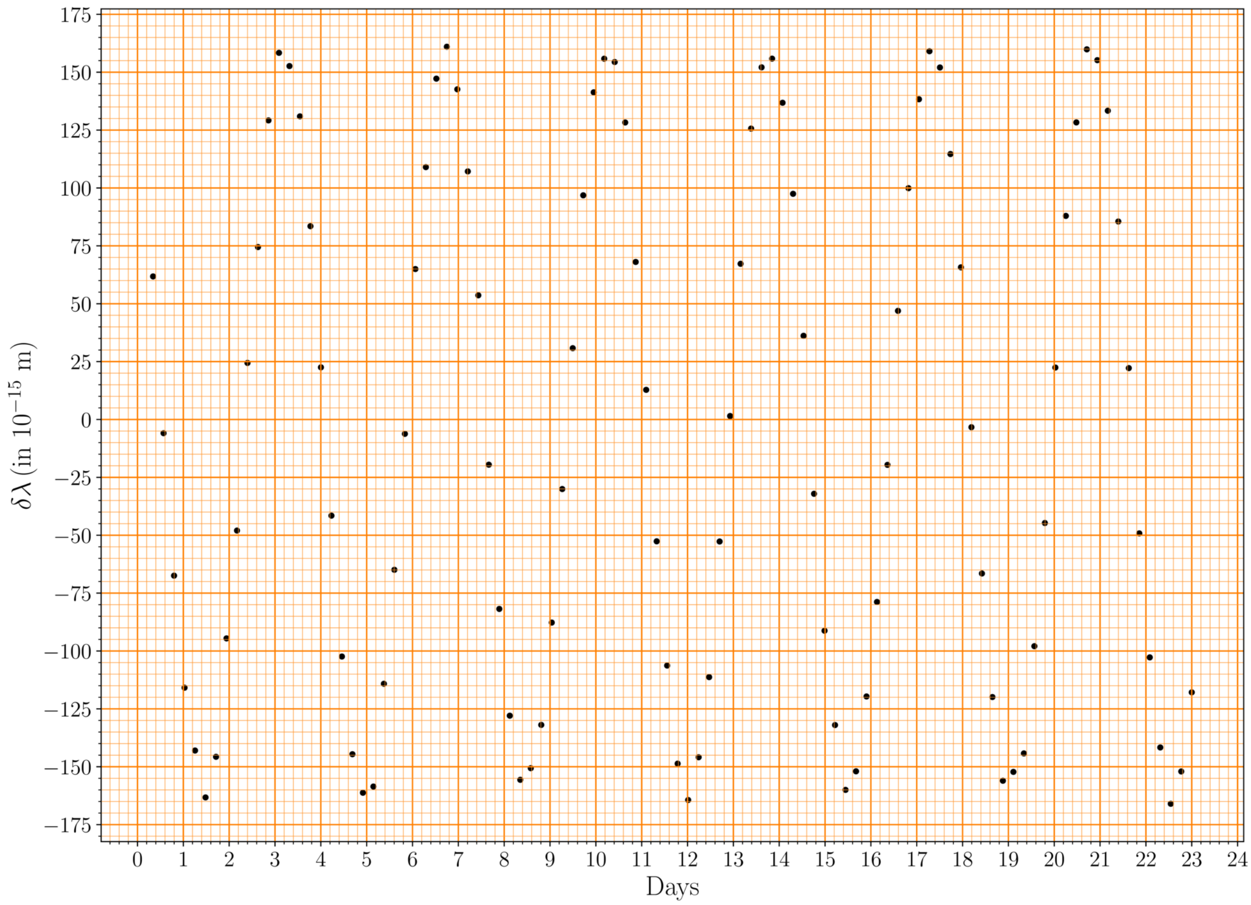
(D01.1) The V-band apparent magnitude of the star S is 7.65 ± 0.03 mag, the parallax is 20.67 ± 0.05 milliarcseconds and the bolometric correction (BC) is −0.0650 mag. Thus the star has a higher bolometric luminosity than its V-band luminosity.

Estimate the mass of the star, M_s (in units of $\mathrm{M}_\odot$), assuming a mass - luminosity (M–L) relation of the form $L \propto M^4$. Also estimate the uncertainty in M_s. You may need $\mathrm{d}\ln x/\mathrm{d}x = 1/x$. **[8]**

Radial Velocity method

The radial velocity method uses the Doppler shift $\delta\lambda \equiv \lambda_\mathrm{obs} - \lambda_0$ between the observed wavelength λ_obs and the rest wavelength λ_0 of a known spectral line to detect an exoplanet and determine its characteristics.

The figure below shows the $\delta\lambda$ for the Fe I line ($\lambda_0 = 543.45 \times 10^{-9}$ m) as a function of time as observed for the SP system.



The radial velocity semi-amplitude K is defined as $K \equiv (v_{\mathrm{r,\,max}} - v_{\mathrm{r,\,min}})/2$ where $v_{\mathrm{r,\,max}}$ and $v_{\mathrm{r,\,min}}$ are the maximum and minimum radial velocities, respectively. For a circular planetary orbit the semi-amplitude K can be written as:

$$K = \left(\frac{2\pi G}{T}\right)^{1/3} \frac{M_{\mathrm{p}} \sin i}{(M_{\mathrm{p}} + M_{\mathrm{s}})^{2/3}}$$

where T is the period, i is the inclination of the planetary orbit (angle between the normal to the orbital plane of the planet and the line of sight of the observer), M_{p} and M_{s} are the masses of the planet and the star, respectively.

(D01.2) Use the above graph given in the Summary Answersheet (rotated by 90°) to answer the following.

(D01.2a) Draw a smooth curve associated with the observed data shown in the graph. **[2]**

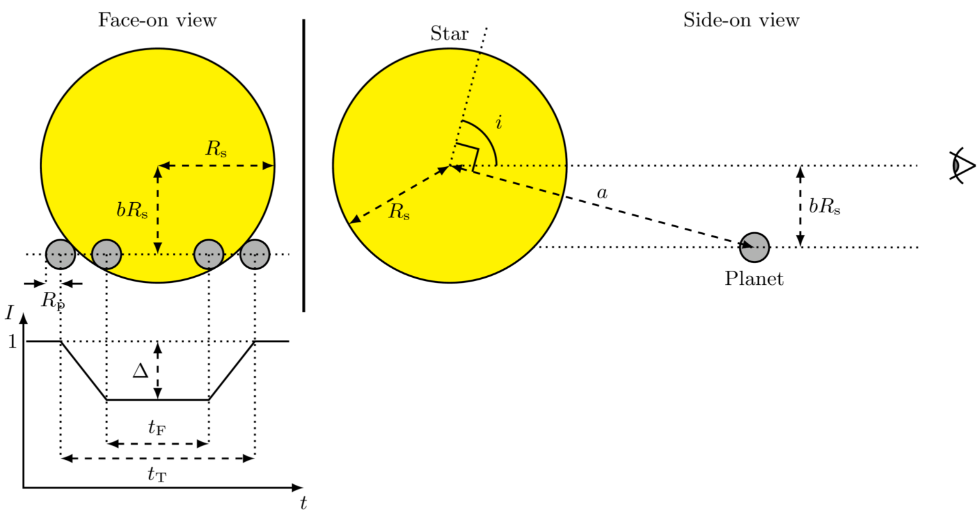
(D01.2b) Select appropriate points on your drawn curve and use suitable methods to determine T and K along with respective uncertainties. All data points used for the calculation of T and K must be shown in the table in the Summary Answersheet. Use the rest of the Table to show your intermediate calculations, as needed, with appropriate headers. **[11]**

(D01.2c) Find the minimum mass of the planet $M_{\mathrm{p,\,min}}$ (in $\mathrm{M}_{\odot}$), and its corresponding uncertainty assuming $M_{\mathrm{p}} \ll M_{\mathrm{s}}$. **[5]**

(D01.2d) Using the value of $M_{\mathrm{p,\,min}}$ estimated in part (D01.2c), calculate the minimum value of the semi-major axis of the planet's orbit, a_{min}, in au and its uncertainty. **[4]**

Transit method (without limb darkening)

The schematic diagram of a planet transit (not drawn to scale) is shown below. Initially, we shall assume the stellar disc to have a uniform average intensity with some intrinsic noise due to the star itself.



The light curve of the normalised intensity, I, as a function of time t is shown in the schematic diagram of the transit above. The average stellar intensity outside the transit is taken as unity. The maximum decrease in the intensity is given by Δ in the normalized light curve. For a uniformly bright stellar disc, the radius of the planet, R_{p}, is related to Δ as

$$\left(\frac{R_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^2 = \Delta,$$

where R_{s} is the radius of the star.

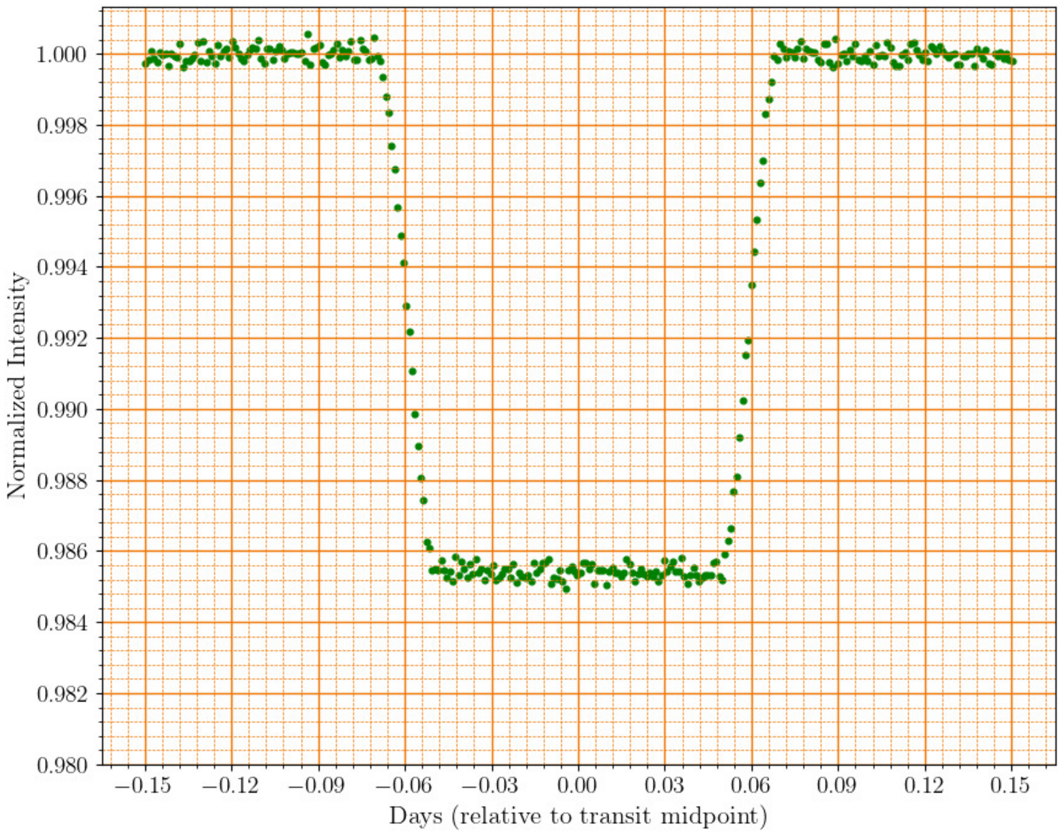
The total duration of transit (when part or all of the planet covers the stellar disc) is given by t_{T}, while t_{F} gives the duration when the planet is fully in front of the stellar disc. The "impact parameter" b is the projected distance between the planet and centre of the stellar disc at the mid-point of the transit, in units of the stellar radius, R_{s}.

For a nearly edge-on star-planet orbit, the impact parameter is given by the formula

$$b = \left[\frac{(1-\sqrt{\Delta})^2 - (t_{\rm F}/t_{\rm T})^2(1+\sqrt{\Delta})^2}{1-(t_{\rm F}/t_{\rm T})^2} \right]^{1/2}$$

(D01.3) For the SP system, the stellar radius is known to be $R_{\rm s} = 1.20\ {\rm R}_\odot$, and the transit of the planet is indeed visible. Using the minimum orbital radius, $a_{\rm min}$, estimated in part (D01.2d), find the minimum value, $i_{\rm min}$, of the inclination angle. **[3]**

Assuming a stellar disc of uniform brightness, the transit light curve would look like as shown below.



(D01.4) Using the given light curve answer the following questions. For your reference the above light curve is also given in the Summary Answersheet.

(D01.4a) Estimate the values of $t_{\rm T}$ and $t_{\rm F}$ in days by marking appropriate readings on the graph. **[3]**

(D01.4b) Estimate the mean value of Δ by marking appropriate readings on the graph and hence find $R_{\rm p}$ in units of ${\rm R}_\odot$. **[2]**

(D01.4c) Determine the value of i in degrees assuming the orbital radius to be $a_{\rm min}$. **[2]**

Introducing limb darkening

So far we have assumed the stellar disc to be uniformly bright. In reality, the observed brightness of the stellar disc is not uniform due to "limb darkening" — an optical effect where the central part of the stellar disc appears brighter than the edge, or the "limb".

The limb darkening effect can be measured by the relative intensity $J(\theta) \equiv \frac{I(\theta)}{I(0)}$, where θ is the angle between the normal to the stellar surface at a point and the line joining the observer to that point, $I(\theta)$ is the observed intensity of the stellar disc at that point and $I(0)$ is the intensity at the centre of the stellar disc. For a distant observer, θ varies from $\theta = 0^\circ$ (centre of the disk) to $\theta \approx 90°$ (edge of the disc).

(D01.5) The table below gives measured $J(\theta)$ at a certain wavelength for the Sun. We shall assume that the same limb darkening profile holds for the star S.

θ	$J(\theta)$	θ	$J(\theta)$	θ	$J(\theta)$	θ	$J(\theta)$
0°	1.000	20°	0.971	40°	0.883	70°	0.595
10°	0.994	25°	0.950	50°	0.794	80°	0.475
15°	0.984	30°	0.943	60°	0.724	90°	0.312

The limb darkening profile can be modelled by a quadratic formula:

$$J(\theta) = 1 - a_1(1 - \cos\theta) - a_2(1 - \cos\theta)^2,$$

where a_1 and a_2 are two constants.

We shall estimate the unknown coefficients a_1 and a_2 from the given data by making a plot with suitable variables.

(D01.5a) Choose a pair of variables (x_1, y_1) which are suitable functions of θ and J, which you will plot along x and y axes respectively, in order to determine a_1 and a_2. Write the expressions for x_1 and y_1. **[2]**

If you need to define additional variables for additional plots, define them as (x_2, y_2), etc.

(D01.5b) Tabulate the values necessary for your plots. **[4]**

(D01.5c) Plot the newly defined variables on the given graph paper (mark your graph as "D01.5c"). **[7]**

(D01.5d) Obtain a_1 and a_2 from the plot. Uncertainties on the values are not needed. **[7]**

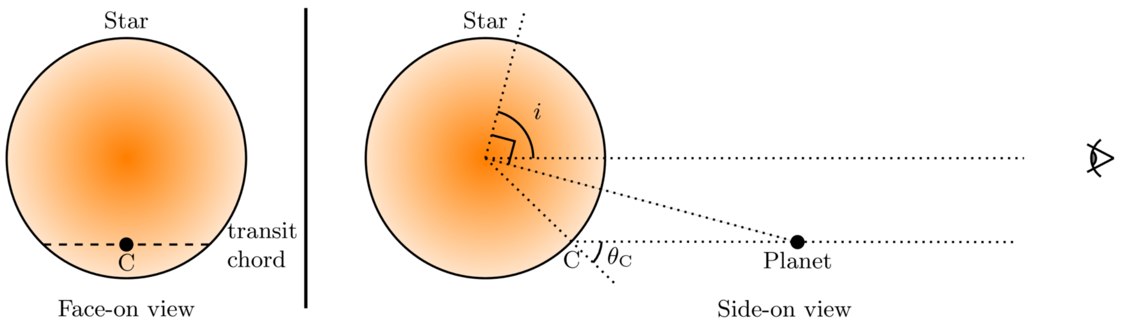
Transit in the presence of limb darkening

Now, we consider planetary transits across a limb-darkened stellar disc. In the presence of limb darkening, which we shall model by the quadratic formula of $J(\theta)$ given above, the average observed intensity of the entire stellar disc (without any transit), $\langle I \rangle$, is given by:

$$\langle I \rangle = \left(1 - \frac{a_1}{3} - \frac{a_2}{6}\right) I(0)$$

Further, the dip in the light caused by the transiting planet now depends not only on the relative size of the planet and the star, $\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)$, but also on the intensity profile of the stellar disc along the transit chord, which in turn, depends on the angle of inclination, i.

The schematic diagram below (not drawn to scale) shows the configuration. Note that the brighter part of the star is shown in a darker shade, while the planet is shown as a black dot.



Here the relation between $\left(\frac{R_p}{R_s}\right)$ and the measured Δ from the light curve is

$$\Delta = \frac{I(\theta_C)}{\langle I \rangle}\left(\frac{R_p}{R_s}\right)^2,$$

where $I(\theta_C)$ is the intensity of the stellar disc at the midpoint of the transit chord (point C in the figure above), and θ_C is the angle between the line of sight and the normal to the surface at that point. From the above it is obvious that for a given star, the same value of Δ can be produced by many combinations of the planet size, R_p, and the inclination angle i.

(D01.6) It is possible to uniquely determine both R_p and i by using data from transit light curves at two wavelengths, say, λ_B (blue) and λ_R (red). The limb darkening coefficients for these two wavelengths are given below:

Wavelength	a_1	a_2
λ_B	0.82	0.05
λ_R	0.24	0.20

(D01.6a) Choose the correct statement among the following that describes the relation between the maximum depth of the transit Δ for λ_B and the inclination angle (i) of the orbit and tick it (✔) in the Summary Answersheet. **[2]**

A. Δ increases with decreasing i.
B. Δ decreases with decreasing i.
C. Δ is independent of i.

(D01.6b) The maximum depth of the transit (Δ) for the "SP system" was measured to be 0.0182 and 0.0159 for λ_B and λ_R, respectively. **[4]**

Draw schematic transit light curves for both λ_B and λ_R on the given grid and label the curves by "B" and "R", respectively. Assume that the total transit duration is same for both wavelengths. The curves need not be to scale, but should represent the shapes of the light curves correctly.

(D01.7) We will use a graphical method to find the values of R_p and i for the SP system using the measurements of Δ at λ_B and λ_R.

(D01.7a) Write an appropriate expression connecting the relevant variables that are to be plotted. (Hint: You may consider i or b, and R_p, among the relevant variables.) **[6]**

(D01.7b) Tabulate the appropriate quantities that are to be plotted. **[5]**

(D01.7c) Draw a suitable graph and mark it as "D01.7c". **[7]**

(D01.7d) Estimate the values of R_p (in $R_\odot$) and i (in degrees) from the graph. **[4]**

(D01.8) Based on the results obtained in this problem, indicate whether the planet P is "ROCKY" or "GASEOUS" by ticking (✓) the appropriate box in the Summary Answersheet. **[2]**

(D02) Predicting arrival times of coronal mass ejections on Earth **[60 marks]**

The Sun occasionally releases magnetised plasma, called coronal mass ejections (CMEs), that originate from the surface of the Sun and propagate outwards. Accurate prediction of their arrival times at Earth is crucial for understanding and mitigating their potential effects on satellites orbiting the Earth. In this problem, we aim to predict the arrival times of CMEs by developing an empirical model, using the data of 10 CMEs. Throughout this problem, the distance between the Sun's surface and the Earth is taken to be $214\ \mathrm{R}_\odot$.

Further, assume that the Sun is not rotating. Due to electromagnetic, gravitational, and drag forces, CMEs experience a variable acceleration throughout their propagation. In the first two parts of this problem, we assume that the region between the Sun and the Earth is a vacuum.

CMEs through vacuum.

(D02.1) The initial velocity, u, at the solar surface ($= 1\ \mathrm{R}_\odot$), the final velocity, v, upon reaching Earth, and the time to arrive at Earth after leaving the surface of the Sun, τ (in hours), are given for 10 CMEs in the following table.

CME	u	v	τ
Name	(km s^{-1})	(km s^{-1})	(h)
CME-A	804	470	74.5
CME-B	247	360	127.5
CME-C	523	396	103.5
CME-D	830	415	71.0
CME-E	665	400	104.5
CME-F	347	350	101.5
CME-G	446	375	99.5
CME-H	155	360	97.0
CME-I	1016	515	67.0
CME-J	683	410	54.0

(D02.1a) Calculate the average acceleration, a, for each CME in m s^{-2}. **[3]**

(D02.1b) We assume an empirical model for the acceleration of a CME, a_{model}, which depends on its initial velocity u as, $a_{\mathrm{model}} = m\left(\frac{u}{u_0}\right) + \alpha$; where, a_{model} is expressed in m s^{-2}, u is expressed in km s^{-1} and $u_0 = 1.00 \times 10^3$ km s^{-1}.

Determine the constants m and α and their associated uncertainties using an appropriate graph (mark your graph as "D02.1b"). **[15]**

(D02.1c) For each CME, tabulate a_{model} in m s^{-2}. Hence calculate the root-mean-square (rms) deviation of accelerations, δa_{rms}, between the calculated acceleration, a, and the model values, a_{model}. **[4]**

(D02.2) We consider two other CMEs: CME-1 and CME-2, with initial velocities, u =1044 km s^{-1} and 273 km s^{-1}, respectively.

(D02.2a) Using the empirical model obtained in (D02.1b), calculate the predicted arrival times at Earth, $\tau_{1,\,\mathrm{m}}$ and $\tau_{2,\,\mathrm{m}}$ (in hours), for CME-1 and CME-2, respectively. **[4]**

(D02.2b) The observed arrival times at Earth of CME-1 and CME-2 are 46.0 h and 74.5 h, respectively. The empirical model is considered to be VALID for a particular CME, if its predicted arrival time is within 20% of its observed arrival time; otherwise, it is NOT VALID. Indicate the validity of the model for each CME by ticking (✓) the appropriate box in the Summary Answersheet. **[2]**

CMEs in presence of solar wind

In reality, the space between the Sun and the Earth is permeated with the solar wind, which exerts a drag force on the CMEs. This drag force can either decelerate or accelerate a CME, depending on the CME's velocity relative to that of the solar wind. To account for the solar wind's influence, we will use a "drag-only" model for distances $R_{\text{obs}}(t) \geq R_0$, where R_0 is the distance beyond which the drag force becomes the dominant force that affects the CME's motion.

As determined from the "drag-only" model, the distance from the surface of the Sun, $R_{\text{D}}(t)$, and the velocity, $V_{\text{D}}(t)$, of a CME in this model is given by

$$R_{\text{D}}(t) = \frac{S}{\gamma} \ln\left[1 + S\gamma(V_0 - V_{\text{s}})(t - t_0)\right] + V_{\text{s}}(t - t_0) + R_0$$

$$V_{\text{D}}(t) = \frac{V_0 - V_{\text{s}}}{1 + S\gamma(V_0 - V_{\text{s}})(t - t_0)} + V_{\text{s}}$$

where, $\gamma = 2 \times 10^{-8}$ km^{-1}, V_{s} is the constant speed of the solar wind, R_0 and V_0 are the distance and velocity at time t_0, and S is the sign factor. $S = 1$ if $V_0 > V_{\text{s}}$; $S = -1$ if $V_0 \leq V_{\text{s}}$.

(D02.3) The tables below show the observed radial distance from the surface of the Sun, $R_{\text{obs}}(t)$ (measured in $R_\odot$), as a function of time t (in hours), for two CMEs: CME-3 and CME-4. The last data point in each table (D5 and P8, respectively) corresponds to the arrival time of the respective CME at Earth. For this part, assume $V_{\text{s}} = 330$ km s^{-1}.

CME-3		
Data point	t (in h)	$R_{\text{obs}}(t)$ (in $R_\odot$)
D1	0.200	6.36
D2	0.480	7.99
D3	1.22	11.99
D4	1.49	13.51
D5	58.05	214

CME-4		
Data point	t (in h)	$R_{\text{obs}}(t)$ (in $R_\odot$)
P1	1.00	4.00
P2	3.00	6.00
P3	4.00	9.00
P4	5.00	11.0
P5	21.0	43.0
P6	50.0	100
P7	85.0	170
P8	111	214

We shall evaluate if the "drag-only" model satisfactorily predicts the arrival times of these CMEs. To use this model, an appropriate choice of t_0 and corresponding R_0 and V_0 needs to be made.

(D02.3a) For CME-3, consider the following two cases: **[6]**
(C1) t_0 taken to be the midpoint of the interval D1 – D2
(C2) t_0 taken to be the midpoint of the interval D3 – D4
Assume the velocity remains constant in each specific interval D1–D2 and D3–D4, but may differ between the two intervals.

Using t_0, R_0, and V_0, calculate the difference between the observed and the predicted radial distance $\delta R_\mathrm{D} \equiv R_\mathrm{obs}(t) - R_\mathrm{D}(t)$ in units of $\mathrm{R}_\odot$ at $t = 58.05$ h, for each of the two cases.

(D02.3b) Evaluate $R_\mathrm{D}(t)$ at points P5, P6, P7, and P8 (between the Sun and the Earth) for CME-4 for the following two cases, adopting the procedure similar to (D02.3a): **[4]**

(C3) t_0 taken to be the midpoint of the interval P1 – P2
(C4) t_0 taken to be the midpoint of the interval P3 – P4.

(D02.3c) Plot $R_\mathrm{D}(t)$ (in $\mathrm{R}_\odot$) vs t (in hours) for the two cases (C3 and C4) for CME-4 at points P5, P6, P7, and P8 (marking your graph as "D02.3c"). On the same graph, draw smooth curves of $R_\mathrm{D}(t)$ for the two cases mentioned above. For this part, take the range of the x axis from 0 to 180 hr. **[10]**

(D02.3d) Using the graph, estimate the absolute difference, $|\delta\tau|$, between the actual arrival time of CME-4 at the Earth and its time of arrival predicted by the drag-only model, for each of the cases C3 and C4. **[4]**

(D02.3e) Indicate whether the following statement is TRUE or FALSE by ticking (✓) the appropriate box in the Summary Answersheet (no written justification needed): **[1]**
"The drag forces exerted by the solar wind on CMEs become dominant for CME-3 at an earlier time compared to CME-4".

(D02.4) Consider drag as the dominant force acting on 10 CMEs in part D02.1. Assume that the "drag-only" model is applicable from and beyond the surface of the Sun ($R_0 = 1\,\mathrm{R}_\odot$), for all CMEs. Estimate and tabulate the solar wind speed V_s in km s^{-1} for each CME. Further, estimate the average solar wind speed $V_{\mathrm{s,\,avg}}$ for all 10 CMEs. **[7]**