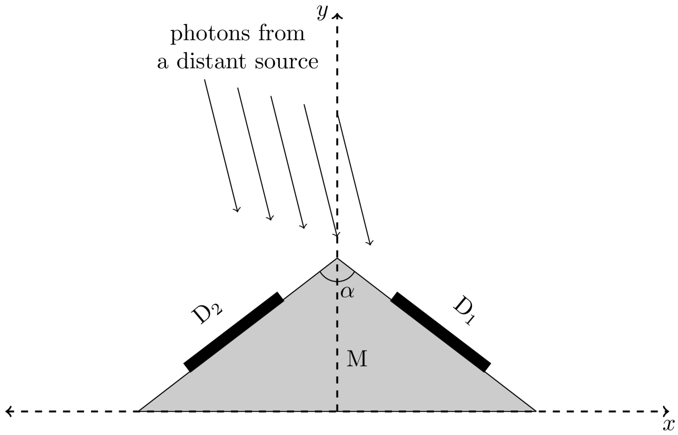
(T01) **Daksha Mission** **[10 marks]**

"Daksha" is a proposed Indian mission consisting of two satellites S_1 and S_2 orbiting the Earth in the same circular orbit of radius $r = 7000\,\mathrm{km}$ but with 180° phase difference. These satellites observe the universe in high energy domain (X-rays and γ-rays). Each of the satellites of Daksha uses several flat, rectangular detectors.

To understand how to localize a source in the sky, we shall use a simplified model of the Daksha mission. Assume that S_1 has only two identical detectors D_1 and D_2, each of area $A = 0.50\,\mathrm{m}^2$, attached to an opaque mount M as shown in the figure below. The detectors lie symmetrically around the y axis in planes perpendicular to the x-y plane and make an angle $\alpha = 120°$ with each other.



(T01.1) When observing a distant source located in the x-y plane, detector D_1 records a power $P_1 = 2.70 \times 10^{-10}\,\mathrm{J\,s^{-1}}$ and detector D_2 records a power $P_2 = 4.70 \times 10^{-10}\,\mathrm{J\,s^{-1}}$. Estimate the angle η made by the position vector of the source with the positive y-axis, with counter-clockwise angle from the positive y axis being considered positive. **5**

Solution:

Since the two detectors make an angle with each other, photons from the source will not be incident on the detectors at the same angle. Using this the source can localized.

Let $\widehat{n}_1$ be the unit vector perpendicular to the plane of D_1 and $\widehat{n_2}$ be the unit vector perpendicular to the plane of D_2. We have,

$$\widehat{n_1} = \frac{1}{2}\widehat{x} + \frac{\sqrt{3}}{2}\widehat{y}$$ 0.5

$$\widehat{n_2} = -\frac{1}{2}\widehat{x} + \frac{\sqrt{3}}{2}\widehat{y}$$ 0.5

Let $\widehat{s} = s_x\widehat{x} + s_y\widehat{y}$ be the unit vector in the direction of the source and F be the flux of the source. The effective area of capture for a detector will be

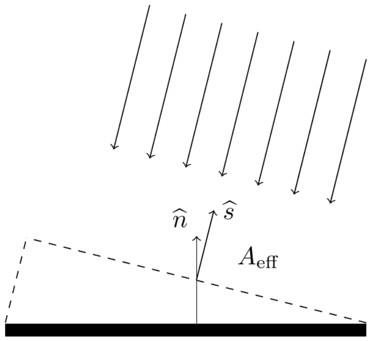
$$A_{\text{eff}} = A(|\widehat{s} \cdot \widehat{n}|)$$ 0.5

Hence, the power received is,

$$P = AF(|\widehat{s} \cdot \widehat{n}|)$$ 1.0

and therefore,

$$P_1 = AF(|\widehat{s} \cdot \widehat{n_1}|)$$
$$P_2 = AF(|\widehat{s} \cdot \widehat{n_2}|)$$



Hence,

$$\widehat{s} \cdot \widehat{n_1} = \frac{1}{2}s_x + \frac{\sqrt{3}}{2}s_y = \frac{2.70 \times 10^{-10}}{AF}$$
$$\widehat{s} \cdot \widehat{n_2} = -\frac{1}{2}s_x + \frac{\sqrt{3}}{2}s_y = \frac{4.70 \times 10^{-10}}{AF}$$

Solving,

$$s_x = -\frac{2.00 \times 10^{-10}}{AF}$$ 0.5
$$s_y = \frac{7.40 \times 10^{-10}}{\sqrt{3}AF}$$ 0.5

Hence, the angle the direction vector makes with respect to the positive y-axis is

$$\eta = \tan^{-1}\left(-\frac{s_x}{s_y}\right)$$ 1.0

Then, $\boxed{\eta = 25.1° = 0.438\,\text{rad}}$. 0.5

Half credit lower limit	Full credit range	Half credit upper limit
–	24.9 to 25.3	–

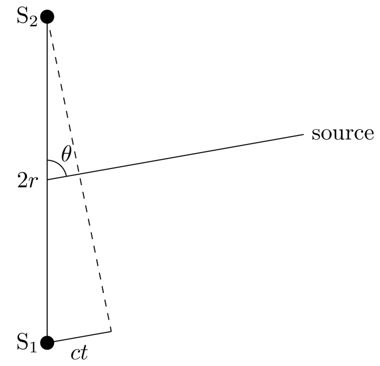
The sign on η carries 0.5 marks.

Consider a single pulse from a distant source (not necessarily in the x-y plane) recorded by both satellites (S_1 and S_2) of Daksha. The times of the peaks of the pulses recorded by S_1 and S_2 are t_1 and t_2, respectively.

(T01.2) If $t_1 - t_2$ was measured to be 10.0 ± 0.1 ms then determine the fraction, f, of the celestial sphere where the source might lie. **5**

Solution:

The time delay occurs because the satellites S_1 and S_2 are located at different distances from the source. Light takes longer time to reach the satellite that is farther (here, say S_1). Let $\Delta t = t_1 - t_2$.



If the direction of the source makes an angle θ (see the figure above) with the line joining telescopes S$_1$ and S$_2$ we have the path difference (Δd) given as,

$$\Delta d = c\Delta t$$ 0.5

From the figure above the path difference can be written as,

$$\Delta d = 2r\cos\theta$$ 1.0

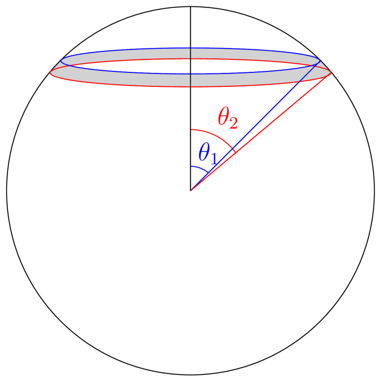
Therefore,

$$\cos\theta = \frac{c\Delta t}{2r}$$

Since the measurement of Δt has some uncertainty, that is $\Delta t \in [9.9, 10.1]$ms, we get a range $[\theta_1, \theta_2]$ for the possible values of θ.

$$\theta_1 = \cos^{-1}\left(\frac{c(10.1\,\text{ms})}{2r}\right) = 77.51^\circ$$ 0.5

$$\theta_2 = \cos^{-1}\left(\frac{c(9.9\,\text{ms})}{2r}\right) = 77.76^\circ$$ 0.5



The source may lie anywhere in the annulus bounded by the red and blue circles on the unit sphere above, whose area is given by

$$2\pi(\cos\theta_1 - \cos\theta_2)$$ 1.5

Hence the fraction of sky where the source might lie is

$$f = \frac{1}{2}(\cos\theta_1 - \cos\theta_2)$$ 0.5

$$\boxed{f = 2.13 \times 10^{-3}}$$ 0.5

Half credit lower limit	Full credit range	Half credit upper limit
–	2.12×10^{-3} to 2.15×10^{-3}	–

(T02) **Makar-Sankranti** **[10 marks]**

The festival of "Makar-Sankranti" is celebrated in India when the Sun appears to enter the zodiacal region of Capricorn (Makar = Capricorn, Sankranti = Entry) as seen from the Earth. It is currently celebrated around 14 January every year. Many years ago this festival also coincided with the Winter Solstice in the northern hemisphere which we assume to take place on 21 December.

(T02.1) Based on the information above, find the year, y_c, when the celebration of this festival last coincided with the Winter Solstice in the northern hemisphere. **3**

Solution:

It is given that currently the Winter Solstice happens on 21 December which is 24 days before the Sun's entry in Capricorn, on 14 January. 0.5

The Sun moves $360°/365.2564\,\text{d} = 0.9856°$ along the Ecliptic, per day.

This corresponds to an angular shift of $0.9856° \times 24 = 23.6544°$ from the Vernal Equinox to entry in Capricorn. 1.0

The Vernal Equinox precesses at a rate of 1° every 71.6 years (given in the Data Sheet) in the direction of decreasing RA.

Hence, to cover 23.6544° it will need $\approx 1694\,\text{yr}$. 1.0

Hence, the celebration of this festival last coincided with the Winter Solstice in the year $2025 - 1694 = 331$.

$$y_c = 331\text{ CE}$$ 0.5

Length of year taken as 365 days: full credit.

Approximate motion of Sun taken as 1° per day: full credit.

Half credit lower limit	Full credit range	Half credit upper limit
	300 CE to 340 CE	

(T02.2) If the Sun appeared to enter the zodiacal region of Capricorn at a local time of 11:50:13 hrs on 14 January 2006 in Mumbai, calculate the date, D_{enter}, and local time, t_{enter}, of its entry in Capricorn in the year 2013. **3**

Solution:

We need to consider the sidereal year which is 365 d 6 h 9 min 13 s. Accounting for two leap years (2008 and 2012) in this interval, the date and time of entry in Capricorn will be as follows: 1.0

Year	Time of Entry	Date
2006	11:50:13	14
2007	17:59:26	14
2008	00:08:39	15
2009	06:17:52	14

Year	Time of Entry	Date
2010	12:27:05	14
2011	18:36:18	14
2012	00:45:31	15
2013	06:54:44	14

Hence, in the year 2013, the Sun enters Capricorn on

$$D_{\text{enter}} = 14\text{ January}$$ 1.0

$$t_{\text{enter}} = 06{:}54{:}44\text{ hrs.}$$ 1.0

Error in min or sec in final answer for t_{enter}: penalty of 0.5

(T02.3) Makar-Sankranti festival is celebrated at a given place on the day of the first sunset in the zodiacal region of Capricorn. You may assume that the local sunset time for Mumbai **4** in January is 18:30:00 hrs.

Indicate the date of celebration of the festival on every year between 2006 and 2013 (by ticking (✓) the respective box in the table given in the Summary Answersheet).

Solution:
Using the table in earlier part:

- Shift forward by one day due to crossing midnight - 2008
- Shift back by one day due to leap year - 2009
- Shift forward by one day due to sunset time - 2011
- Shift back by one day due to leap year - 2013

Year	14 Jan	15 Jan	16 Jan
2006	✓		
2007	✓		
2008		✓	
2009	✓		

Year	14 Jan	15 Jan	16 Jan
2010	✓		
2011		✓	
2012		✓	
2013	✓		

4.0

- Each correct tick (✓): 0.5 marks.
- No marks if cell is left black.

(T03) **Gravitational Waves** **[15 marks]**

Orbiting binary black holes generate gravitational waves. Consider two black holes, in our Galaxy with masses $M = 36\,\mathrm{M}_\odot$ and $m = 29\,\mathrm{M}_\odot$, revolving in circular orbits with orbital angular frequency ω around their centre of mass.

(T03.1) Assuming Newtonian gravity, derive an expression for the angular frequency, ω_{ini}, of the black hole orbits at a time, t_{ini}, when the separation between the black holes was 4.0 times the sum of their Schwarzschild radii, in terms of only M, m, and physical constants. Calculate the value of ω_{ini} (in $\mathrm{rad\,s^{-1}}$). **5**

Solution:

Assuming Newtonian gravity, for black hole masses M and m at distances R and r, respectively, from their centre of mass,

$$\omega^2 r = \frac{GM}{(R+r)^2}$$

$$\omega = \left[\frac{GM}{r(R+r)^2}\right]^{1/2}$$ 1.0

Since $MR = mr$, we can rewrite this as

$$\omega = \left[\frac{G(M+m)}{(R+r)^3}\right]^{1/2}$$ 1.0

Let R_s and r_s be the respective Schwarzschild radii of the black holes of mass M and m. Expressing the separation $(r+R) = 4.0 \times (r_\mathrm{s} + R_\mathrm{s})$ at t_{ini}, we obtain,

$$\omega_{\text{ini}} = \left[\frac{G(M+m)}{4^3(R_\mathrm{s}+r_\mathrm{s})^3}\right]^{1/2} = \left[\frac{c^6 G(M+m)}{8 \times 4^3 G^3 (M+m)^3}\right]^{1/2} = \frac{c^3}{16\sqrt{2}G(M+m)}$$

$$\boxed{\omega_{\text{ini}} = \frac{c^3}{16\sqrt{2}G(M+m)}}$$ 2.0

Substituting the values of the masses,

$\boxed{\text{value of } \omega_{\text{ini}} = 1.4 \times 10^2\,\mathrm{rad\,s^{-1}}}$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$130\,\mathrm{rad\,s^{-1}}$	$136\,\mathrm{rad\,s^{-1}}$ to $140\,\mathrm{rad\,s^{-1}}$	—

- If constants are taken as given in table, answer is $138.084\,\mathrm{rad\,s^{-1}}$.
- If constants are taken up to 2 significant digits, answer is $136.996\,\mathrm{rad\,s^{-1}}$.
- if constants are taken up to 1 significant digit, answer is $131.126\,\mathrm{rad\,s^{-1}}$.

(T03.2) In general relativity, black holes in orbit emit gravitational waves with frequency f_{GW}, such that $2\pi f_{\text{GW}} = \omega_{\text{GW}} = 2\omega$. This shrinks the black hole orbits, which in turn increases f_{GW}. The rate of change of f_{GW} is **4**

$$\frac{df_{\text{GW}}}{dt} = \frac{96\pi^{8/3}}{5} G^{5/3} c^{\beta} M_{\text{chirp}}{}^{\alpha/3} f_{\text{GW}}^{\delta/3},$$

where $M_{\text{chirp}} = \dfrac{(mM)^{3/5}}{(m+M)^{1/5}}$ is called the “chirp mass”.

Find the values of α, β and δ.

Solution:
We use dimensional analysis to calculate the exponents in the above equation.
The dimension of M_{chirp} is [M].
The dimensions of the left and right hand sides of the equation are

$$[\mathrm{T}]^{-2} = [\mathrm{T}]^{-\delta/3}\left([\mathrm{L}]^3[\mathrm{T}]^{-2}[\mathrm{M}]^{-1}\right)^{5/3}\left([\mathrm{L}][\mathrm{T}]^{-1}\right)^{\beta}[\mathrm{M}]^{\alpha/3}$$ 1.0

Equating the dimensions for T, L, and M, we get

$$-2 = -\frac{\delta}{3} - \frac{10}{3} - \beta$$
$$0 = 5 + \beta$$
$$0 = \frac{\alpha}{3} - \frac{5}{3},$$

respectively. Solving, we get $\boxed{\alpha = 5,\ \beta = -5 \text{ and } \delta = 11}$. 3.0

If any parameter values are incorrect,

- 1.0 mark for the correct equation only if it is written explicitly.
- 1.0 mark for every correct parameter value.

(T03.3) Assume that the gravitational waves associated with the event were first detected at time $t_{\text{ini}} = 0$. **6**
Derive an expression for the observed time of black hole merger, t_{merge}, when f_{GW} becomes very large, in terms of ω_{ini}, M_{chirp}, and physical constants only.
Calculate the value of t_{merge}(in seconds).

Solution:
From (T03.2), we get

$$\frac{df_{\text{GW}}}{dt} = \frac{96\pi^{8/3}}{5}\frac{G^{5/3}}{c^5}M_{\text{chirp}}{}^{5/3}f_{\text{GW}}^{11/3}.$$ 1.0

Integrating the equation for f_{GW}, we obtain

$$\int_{\omega_{\text{ini}}/\pi}^{\infty} f_{\text{GW}}^{-11/3}df_{\text{GW}} = \int_0^{t_{\text{merge}}}\frac{96\pi^{8/3}}{5}\frac{G^{5/3}}{c^5}M_{\text{chirp}}{}^{5/3}dt$$ 1.0

$$-\frac{3}{8}f_{\text{GW}}^{-8/3}\Big|_{\omega_{\text{ini}}/\pi}^{\infty} = \frac{96\pi^{8/3}}{5}\frac{G^{5/3}}{c^5}M_{\text{chirp}}{}^{5/3}t_{\text{merge}}$$ 1.0

$$t_{\text{merge}} = \frac{3}{8}\left[\frac{\omega_{\text{ini}}}{\pi}\right]^{-8/3}\frac{5}{96\pi^{8/3}}\frac{c^5}{G^{5/3}}\frac{1}{M_{\text{chirp}}{}^{5/3}}$$
$$= \frac{5}{256}\omega_{\text{ini}}^{-8/3}\frac{c^5}{(GM_{\text{chirp}})^{5/3}}.$$

$$\boxed{t_{\text{merge}} = \frac{5}{256}\omega_{\text{ini}}^{-8/3}\frac{c^5}{(GM_{\text{chirp}})^{5/3}}}$$ 1.0

Answer with correct exponents, but wrong coefficients gets 1.0 mark.

We calculate $M_{\text{chirp}} = \frac{(mM)^{3/5}}{(m+M)^{1/5}} = 28\,\mathrm{M}_{\odot}$. 1.0
Substituting the values of M_{chirp}, and ω_{ini} we get the

value of $t_{merge} = 0.10\,s$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
–	$0.090\,s$ to $0.110\,s$	–

(T04) **Balmer Decrement** **[15 marks]**

Consider a main sequence star surrounded by a nebula. The observed V-band magnitude of the star is 11.315 mag. The ionised region of nebula close to the star emits Hα and Hβ lines; their wavelengths are 0.6563 µm and 0.4861 µm, respectively. The theoretically predicted ratio of fluxes in Hα to Hβ lines is $f_{\mathrm{H}\alpha}/f_{\mathrm{H}\beta} = 2.86$. However, when this radiation passes through the outer portion of the cold dusty nebula, the observed emission fluxes of Hα and Hβ lines are $6.80 \times 10^{-15}\,\mathrm{W\,m^{-2}}$ and $1.06 \times 10^{-15}\,\mathrm{W\,m^{-2}}$, respectively.

The extinction A_λ is a function of wavelength and is expressed as

$$A_\lambda = \kappa(\lambda)E(B-V).$$

Here, $\kappa(\lambda)$ is the extinction curve and $E(B-V)$ denotes the colour excess in the filter bands B and V. The extinction curve (with λ in µm) is given as follows.

$$\kappa(\lambda) = \begin{cases} 2.659 \times \left(-1.857 + \frac{1.040}{\lambda}\right) + R_V, & 0.63 \le \lambda \le 2.20 \\ 2.659 \times \left(-2.156 + \frac{1.509}{\lambda} - \frac{0.198}{\lambda^2} + \frac{0.011}{\lambda^3}\right) + R_V, & 0.12 \le \lambda < 0.63 \end{cases}$$

where, $R_V = A_V/E(B-V) = 3.1$ is the ratio of total-to-selective extinction.

(T04.1) Find the values of $\kappa(\mathrm{H}\alpha)$ and $\kappa(\mathrm{H}\beta)$. **3**

Solution:

For Hα substituting in equation for $\kappa(\lambda)$, we get

$$\kappa(\mathrm{H}\alpha) = 2.659 \times \left(-1.857 + \frac{1.040}{0.6563}\right) + 3.1$$

$$\boxed{\kappa(\mathrm{H}\alpha) = 2.4}$$ 1.5

Similarly, for Hβ we have

$$\kappa(\mathrm{H}\beta) = 2.659 \times \left(-2.156 + \frac{1.509}{0.4861} - \frac{0.198}{0.4861^2} + \frac{0.011}{0.4861^3}\right) + 3.1$$

$$\boxed{\kappa(\mathrm{H}\beta) = 3.6}$$ 1.5

For $\kappa(\mathrm{H}\beta)$

Half credit lower limit	Full credit range	Half credit upper limit
	3.6 to 3.7	

(T04.2) Find the value of the ratio of colour excess $\dfrac{E(\mathrm{H}\beta - \mathrm{H}\alpha)}{E(B-V)}$. **4**

Solution:

$$E(\mathrm{H}\beta - \mathrm{H}\alpha) = A_{\mathrm{H}\beta} - A_{\mathrm{H}\alpha}$$ 2.0

$$= [\kappa(\mathrm{H}\beta) - \kappa(\mathrm{H}\alpha)]\,E(B-V) = 1.270\,E(B-V)$$

$$\boxed{\frac{E(\mathrm{H}\beta - \mathrm{H}\alpha)}{E(B-V)} = 1.2}$$ 2.0

Credit of only 2.0 marks if answer is given without showing earlier expression relating color excess $E(\mathrm{H}\beta - \mathrm{H}\alpha)$ to extinction A.

Half credit lower limit	Full credit range	Half credit upper limit
	1.2 to 1.3	

(T04.3) Estimate the extinction due to nebula, $A_{\mathrm{H}\alpha}$ and $A_{\mathrm{H}\beta}$, at Hα and Hβ wavelengths, respectively. **6**

Solution:

If $f_{\lambda,\mathrm{obs}}$ and $f_{\lambda,\mathrm{em}}$ represent the observed and emitted fluxes at wavelength λ, respectively, then the extinction is given by the equation

$$A_\lambda = -2.5\log\left(\frac{f_{\lambda,\mathrm{obs}}}{f_{\lambda,\mathrm{em}}}\right)$$ 1.0

$$\text{Given, } \frac{f_{\mathrm{H}\alpha,\mathrm{em}}}{f_{\mathrm{H}\beta,\mathrm{em}}} = 2.86$$

$$\begin{aligned} E(\mathrm{H}\beta - \mathrm{H}\alpha) &= A_{\mathrm{H}\beta} - A_{\mathrm{H}\alpha} \\ &= -2.5\log\left(\frac{f_{\mathrm{H}\beta,\mathrm{obs}}}{f_{\mathrm{H}\alpha,\mathrm{obs}}} \times \frac{f_{\mathrm{H}\alpha,\mathrm{em}}}{f_{\mathrm{H}\beta,\mathrm{em}}}\right) \\ &= -2.5\log\left(\frac{1.06\times10^{-15}}{6.80\times10^{-15}} \times 2.86\right) \\ &= 0.88 \end{aligned}$$ 1.5

From extinction curve, we have $E(\mathrm{H}\beta - \mathrm{H}\alpha) = 1.2\,E(B-V)$.
Equating these two, we obtain

$$E(B-V) = \frac{0.88}{1.2} = 0.73$$ 0.5

This gives,

$$A_{\mathrm{H}\alpha} = \kappa(\mathrm{H}\alpha)\,E(B-V) = 2.4\times0.73$$

$$\boxed{A_{\mathrm{H}\alpha} = 1.8\,\mathrm{mag}}$$ 1.5

$$A_{\mathrm{H}\beta} = \kappa(\mathrm{H}\beta)\,E(B-V) = 3.6\times0.73$$

$$\boxed{A_{\mathrm{H}\beta} = 2.6\,\mathrm{mag}}$$ 1.5

Full credit for $E(B-V)$ between 0.69 - 0.73, no credit outside this range. Other values do not change when rounded to 2 significant digits.

For $A_{\mathrm{H}\alpha}$

Half credit lower limit	Full credit range	Half credit upper limit
	1.6 to 1.8	

For $A_{\mathrm{H}\beta}$

Half credit lower limit	Full credit range	Half credit upper limit
	2.5 to 2.6	

(T04.4) Estimate the extinction of the nebula (A_V) and the apparent magnitude of the star in the V band, $m_{\mathrm{V}0}$, in the absence of the nebula. **2**

Solution:

$$A_V = 3.1\times E(B-V) = 3.1\times0.69$$

$$\boxed{A_V = 2.3\,\mathrm{mag}}$$ 1.0

$$m_{\mathrm{V}0} = m_V - A_V = 11.315 - 2.3$$

$$\boxed{m_{\mathrm{V}0} = 9.0\,\mathrm{mag}}$$ 1.0

For A_V

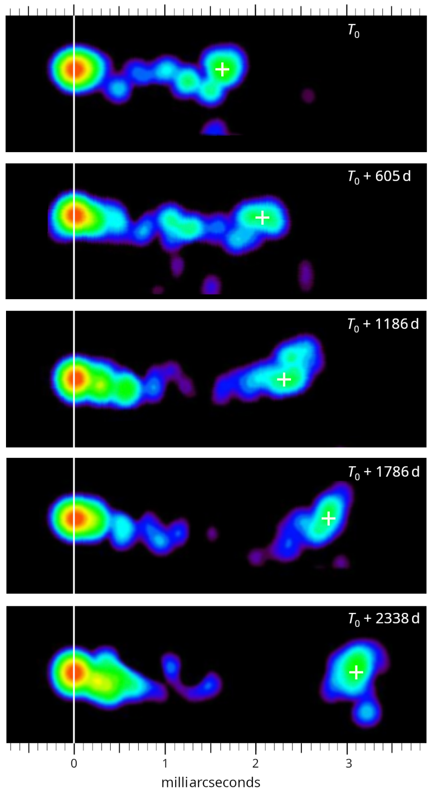
Half credit lower limit	Full credit range	Half credit upper limit
	2.1 mag to 2.3 mag	

For m_{V0}

Half credit lower limit	Full credit range	Half credit upper limit
	9.0 mag to 9.2 mag	

(T05) **Quasars** **[20 marks]**

A quasar is an extremely luminous active galaxy powered by a supermassive black hole that emits relativistic jets. The figure shows a series of panels of radio images of a quasar (with redshift $z = 0.53$, and luminosity distance $D_L = 1.00 \times 10^{10}$ ly) at different times. The "core" aligns with the vertical white line, while a jet, consisting of a "blob" (marked white +), moves away from it over time. Each panel shows the observation time (starting with T_0 for the first image), and the angular scale is indicated at the top and bottom of the figure.



(T05.1) Determine the blob's angular separation, $\phi_{\rm blob}$ (in milliarcsecond), and its transverse distance, $l_{\rm blob}$ (in light-year), from the quasar core for each observation. Then, calculate the blob's apparent velocity in the transverse direction ($v_{\rm app}$) as a fraction of light speed, $\beta_{\rm app}$ ($= v_{\rm app}/c$) by using consecutive observations. Also calculate the average apparent velocity $\beta_{\rm app}^{\rm ave}$ over the entire observation period. **5**

Solution:

$$\text{Angular diameter distance} = D_L/(1+z)^2$$

$$\text{Scale} = \text{Angular diameter distance} \times \frac{\pi}{60 \times 60 \times 180} \text{ly/arcsecond}$$

$$= 20.7 \,\text{ly milliarcsecond}^{-1}.$$ 1.0

Date of observation	$\phi_{\rm blob}$ (milliarcsecond)	$l_{\rm blob}$ (ly)	$\beta_{\rm app}$
T_0	1.6	33	×
$T_0 + 605\,$d	2.1	44	6.6
$T_0 + 1186\,$d	2.3	48	2.5
$T_0 + 1786\,$d	2.8	58	6.1
$T_0 + 2338\,$d	3.1	64	4.0

3.0

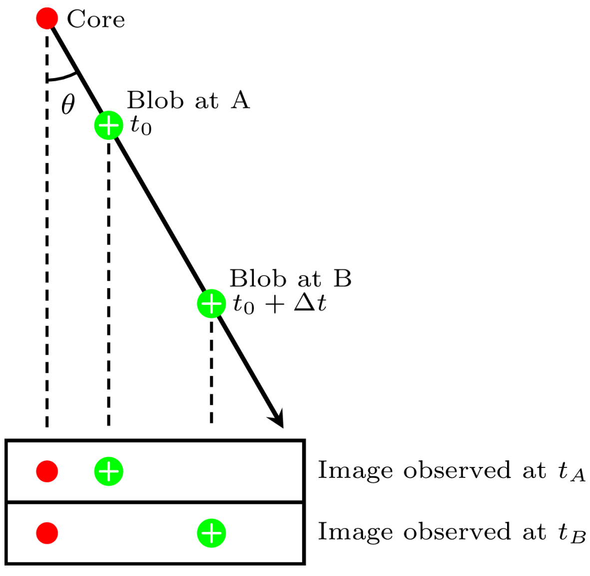
All ϕ_{blob} entries are correct	1.0
3-4 ϕ_{blob} entries are correct	0.5

Similar marking for l_{blob} and β_{app}, but without double penalty
Tolerance in each ϕ_{blob} is ± 0.1 milliarcsecond.
Tolerance in each l_{blob} is ± 2 ly.
Tolerance in each β_{app} is ± 1.2

The average apparent velocity can be calculated by using the distance and time between the first and the last point which yields $\beta_{\text{app}}^{\text{ave}} = 4.8$.

Half credit lower limit	Full credit range	Half credit upper limit
	3.6 to 6.4	

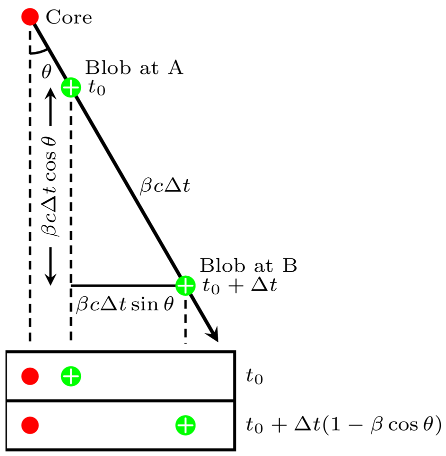
The quasar jet actually moves at a relativistic speed $v \equiv \beta c$, but not necessarily in the plane of the sky; e.g., it makes an angle θ (the "viewing angle") with respect to the line of sight of a distant observer (indicated by the dashed lines), as shown in the sketch below.
For this and all subsequent parts, ignore redshift of the quasar and any relativistic effects.



(T05.2) The light emitted by the blob at two different times t_0 (corresponding to position A) and $t_0 + \Delta t$ (corresponding to position B) reaches the observer at t_{A} and t_{B}, respectively. Thus the observed time difference is $\Delta t_{\text{app}} = t_{\text{B}} - t_{\text{A}}$.

(T05.2a) Find an expression for the ratio $\dfrac{\Delta t_{\text{app}}}{\Delta t}$ in terms of β and θ. **2**

Solution:
Radiation from the core always takes the same time to reach us, but the time taken by radiation from the moving blob in the jet decreases since its distance from us reduces.



As shown in the sketch above, a signal emitted from the moving blob at B covers a distance $(\beta c)\Delta t \cos\theta$ less than a signal emitted at A.
Thus the apparent time difference between the light received by the observer is

$$\Delta t_{\text{app}} = t_{\text{B}} - t_{\text{A}} = \left[t_0 + \Delta t - \frac{(\beta c)\Delta t \cos\theta}{c}\right] - t_0$$

$$= \Delta t(1 - \beta\cos\theta)$$

$$\implies \boxed{\frac{\Delta t_{\text{app}}}{\Delta t} = (1 - \beta\cos\theta)}.$$

2.0

In case the answer is incorrect but the diagram has the correct entries for the sides of the relevant triangle (i.e., components of $\beta c \Delta t$), then we give 1 mark as partial credit.

(T05.2b) Using this ratio, express β_{app} in terms of β and θ. **2**

Solution:
The apparent velocity on the sky is

$$\beta_{\text{app}} = \frac{v_{\text{app}}}{c} = \frac{\beta\ \Delta t\ \sin\theta}{\Delta t_{\text{app}}}$$

$$\boxed{\beta_{\text{app}} = \frac{\beta\ \sin\theta}{1 - \beta\ \cos\theta}}$$

2.0

The resulting formula indicates that the measured apparent velocity of the jet's component is increased for relativistic features and can be higher than speed of light for small values of angle θ.

(T05.3) Motion is called superluminal if the apparent speed exceeds that of light ($\beta_{\text{app}} > 1$), and subluminal if it does not ($\beta_{\text{app}} < 1$).

(T05.3a) For $\beta_{\text{app}} = 1$, plot a smooth curve of β as a function of θ to mark the boundary between subluminal and superluminal motions. Shade the superluminal region in the graph with slanted lines (////). **4**

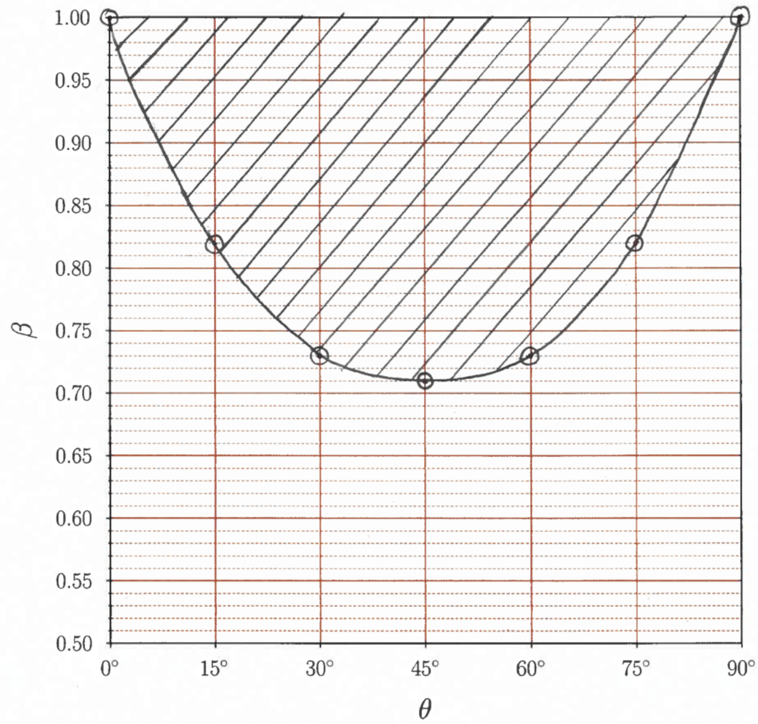
Solution:
For $\beta_{\text{app}} = 1$, we obtain

$$\beta(\sin\theta + \cos\theta) = 1 \implies \beta = \frac{1}{\sin\theta + \cos\theta}$$

Superluminal motion occurs if $\beta_{\text{app}} > 1$; then

$$\beta(\sin\theta + \cos\theta) > 1.$$

This defines the superluminal region which is to be shaded.



Correct values at each of 15°, 30°, 45°, 60° and 75°	2.5
Curve is overall smooth and symmetric	0.5
Correct region is shaded	1.0

(T05.3b) Find the lowest true jet speed ($\beta_{\text{low}} = v_{\text{low}}/c$) for the superluminal motion to occur and also its corresponding viewing angle θ_{low}. **2**

Solution:
It may be seen from graph drawn in (T05.3a) that the minimum value of β for which superluminal motion is apparent occurs at $\theta = 45°$, i.e., $\beta = 1/\sqrt{2}$.

$\beta_{\text{low}} = 1/\sqrt{2} \approx 0.707$, and 1.0

the corresponding orientation angle, $\theta_{\text{low}} = 45°$ 1.0

(T05.4) Find an expression for the maximum viewing angle, θ_{max}, for which a given value of β_{app} will be possible. **2**

Solution:
We know

$$\beta_{\text{app}} = \frac{\beta\sin\theta}{1 - \beta\cos\theta}$$

Thus

$$\beta\left(\frac{\sin\theta + \beta_{\text{app}}\cos\theta}{\beta_{\text{app}}}\right) = 1$$

Since $\beta \leq 1$, we have

$$\frac{\sin\theta + \beta_{\text{app}}\cos\theta}{\beta_{\text{app}}} > 1$$ 1.0

or

$$\beta_{\text{app}}\tan(\theta/2) < 1.$$

Therefore, the maximum viewing angle for a jet with a given β_{app} is

$$\boxed{\theta_{\max} = 2\tan^{-1}\left[\frac{1}{\beta_{\text{app}}}\right]}$$ 1.0

The core of a quasar, its central compact object, exhibits variability in its emission due to internal processes occurring within a causally connected region. The size (= radius) of this region is typically taken to be about five times the Schwarzschild radius of the core.

(T05.5) The core of a certain quasar is found to vary on time scales of about 1 h. Obtain an upper limit, $M_{\text{c, max}}$, on the mass of the central compact object, in units of solar mass. **3**

Solution:

Time-scale δt of any substantial variability from a source cannot be shorter than the light crossing time of the source.

This then gives an upper limit to the size $\lesssim c\,\delta t$, where c is the speed of light.

Therefore,

$$c\delta t \gtrsim 5R_{\text{SC}} = 5 \times \frac{2GM}{c^2}.$$ 1.0

where, M is the mass of the central compact object

Rearranging,

$$M \lesssim \frac{c^3\delta t}{10G}$$

Thus, upper limit on the mass of central compact core of quasar

$$M_{\text{c, max}} = \frac{c^3\delta t}{10G}$$ 1.0

$$= \frac{(2.998\times10^8)^3\times3600}{10\times6.674\times10^{-11}}\,\text{kg} = 7.311\times10^7\,\text{M}_\odot$$

$$\boxed{M_{\text{c, max}} = 7\times10^7\,\text{M}_\odot}$$ 1.0

(T06) **Galactic Rotation** **[20 marks]**

The rotation curve of our Galaxy is determined using line-of-sight velocity measurements of neutral hydrogen (HI) clouds along various Galactic longitudes, observed through the 21 cm HI line. Consider an HI cloud with Galactic longitude l, located at a distance R from the Galactic Centre (GC) and a distance D from the Sun. Consider Sun to be at a distance $R_0 = 8.5$ kpc from the GC. Assume that both the Sun and the HI cloud are in circular orbits around the GC in the Galactic plane, with angular velocities Ω_0 and Ω, and rotational velocities V_0 and V, respectively.

The line-of-sight velocity (V_r) and transverse velocity (V_t) components of the cloud, as observed from the Sun, can be expressed as

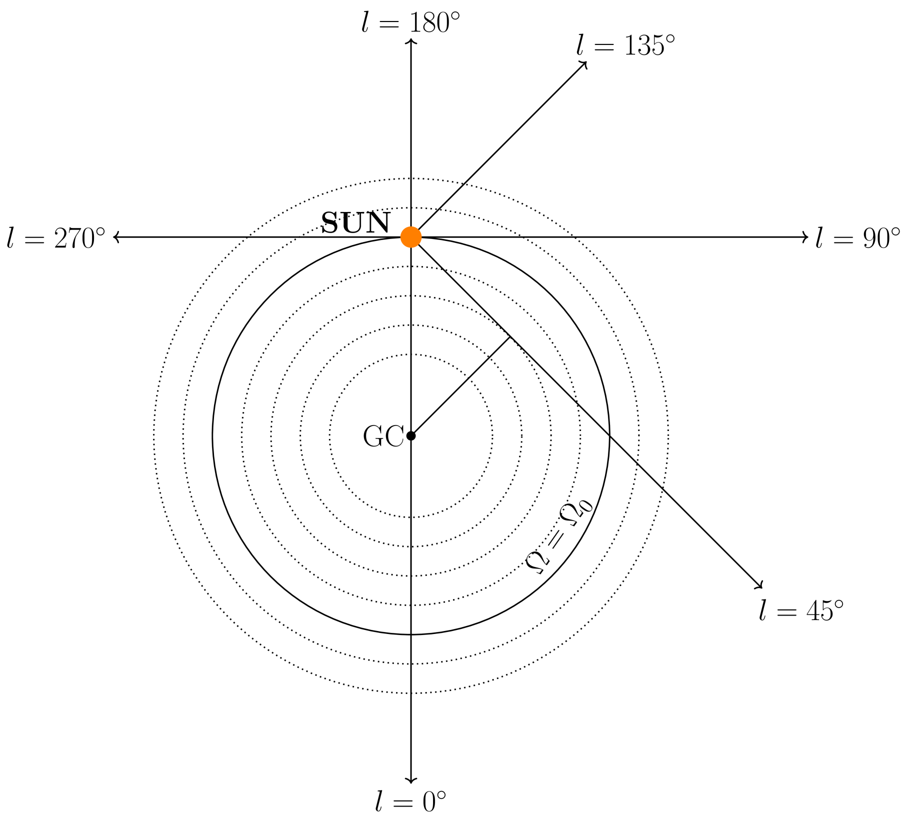
$$V_r = (\Omega - \Omega_0) R_0 \sin l$$

$$V_t = (\Omega - \Omega_0) R_0 \cos l - \Omega D$$

Seen from the North Galactic Pole, the Galactic rotation is clockwise. Throughout this problem, we shall take line-of-sight velocity to be positive when receding and clouds will be treated as point objects.

(T06.1) In the graph provided on the Summary Answersheet, sketch V_r as a function of D from $D = 0$ to $D = 2R_0$ for two lines of sight defined by (i) $l = 45°$ and (ii) $l = 135°$. Label each of your lines/curves with the value of l. **5**

Solution:



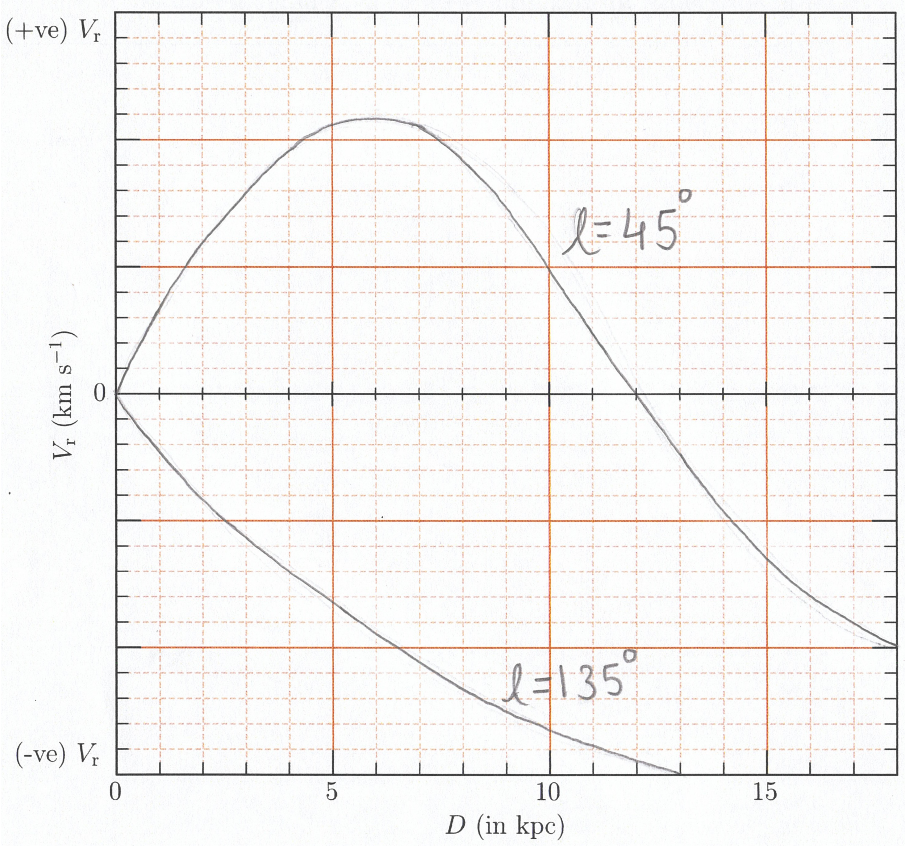
The concept of differential rotation of the Galaxy is used here.

- For $l = 45°$:
  - From Sun to the GC, the quantity $(\Omega - \Omega_0)$ increases, which implies V_r increases.

  - As shown in the figure, the line of sight intersects circular orbits that are at varying distances from the GC.
  - The line of sight is closest to the GC when $D = R_0 \cos l = 6.01\,\text{kpc}$. V_r reaches the maximum value here.
  - Beyond this, $(\Omega - \Omega_0)$ decreases implying V_r decreases.
  - $V_r = 0$ when the line of sight intersects the Sun's orbit. Here, $D = 2R_0 \cos l = 12.02\,\text{kpc}$
  - Beyond the Sun's orbit, $\Omega < \Omega_0$. Vr is negative and its magnitude increases with increasing D.

- $l = 135°$
  - The line of sight lies outside the Sun's orbit. Here, V_r is negative, there is no peak, and its magnitude increases with increasing D.

The plot of the line-of-sight velocity as a function of the distance from the Sun for two values of l is as follows:



For $l = 45°$:

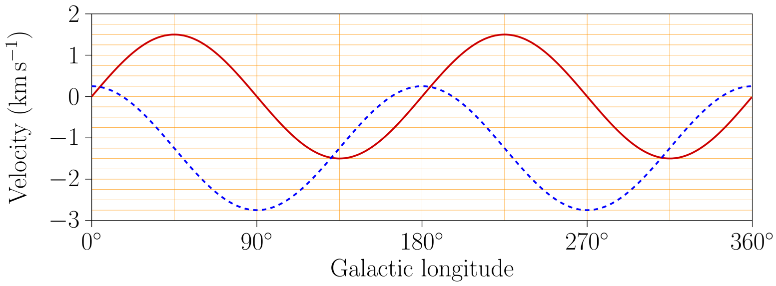
Starts at (0,0)	0.5
Has one and only one positive maximum	0.5
Maximum is at $D = 6\,\text{kpc}$	0.5
Has a zero crossing at $D = 12\,\text{kpc}$	1.0
Is negative throughout beyond zero crossing	0.5

Is smooth throughout	0.5

For $l = 135°$:

Starts at (0,0)	0.5
Is negative throughout without any maxima/minima	0.5
Is smooth throughout	0.5

(T06.2) The graph below shows the average radial (solid, red curve) and transverse (dashed, blue curve) velocity components of stars at a distance of 100 pc from the Sun, plotted as a function of Galactic longitude. **3**



Using the graph, estimate the Sun's orbital period (P) around the GC in mega-years (Myr).

Solution:

Consider $l = 90$,

$V_r = 0$ implying $\Omega = \Omega_0$ 0.5

$V_t = -\Omega_0 D = -2.75\,\mathrm{km\,s^{-1}}$ 0.5

$\Omega_0 = 8.91 \times 10^{-16}\,\mathrm{rad\,s^{-1}}$ 1.0

$$P = \frac{2\pi}{\Omega_0} = \boxed{224\,\mathrm{Myr}}$$ 1.0

Expression of V_r for a chosen l	0.5
Expression of V_t for a chosen l	0.5
Calculation of Ω_0 for a chosen l (acceptable range 8.8 to $9.1 \times 10^{-16}\,\mathrm{rad\,s^{-1}}$)	1.0
Calculation of P for a chosen l	1.0

Half credit lower limit	Full credit range	Half credit upper limit
	218 Myr to 228 Myr	

(T06.3) Jan Oort noted that in the solar neighbourhood ($D \ll R_0$), the difference in angular velocities ($\Omega - \Omega_0$) will be small, and hence, derived the following first order approximation for the line-of-sight and the transverse velocity components:

$$V_r = AD \sin 2l$$

$$V_t = AD \cos 2l + BD$$

where A and B are known as Oort's constants.

Let us consider two cases:

(I) the actual observed rotation curve of the Galaxy, and

(II) the rotation curve is for a hypothetical scenario where the Galaxy is devoid of dark matter and the whole mass of the Galaxy is assumed to be concentrated at its centre.

(T06.3a) Derive expressions for the radial gradient of the rotational velocity at the location of the Sun, $\left.\frac{dV}{dR}\right|_{R=R_0}$, for the two cases. **2**

Solution:

(I) The observed rotation curve is flat near the Sun.

V is constant $\rightarrow$ $\boxed{\left.\frac{dV}{dR}\right|_{R=R_0} = 0}$. 1.0

(II) If the whole mass of the Galaxy is assumed to be concentrated at the centre, then we consider Keplerian motion.

$V = \sqrt{\frac{GM}{R}} \rightarrow \boxed{\left.\frac{dV}{dR}\right|_{R=R_0} = -\frac{1}{2}\frac{V_0}{R_0}}$. 1.0

(T06.3b) Express A and B in terms of V_0, R_0, and the radial gradient of rotational velocity at the location of the Sun, $\left.\frac{dV}{dR}\right|_{R=R_0}$. **8**

Solution:
In the solar neighbourhood, $(\Omega - \Omega_0)$ will be very small. Using the approximation

$f(x) \simeq f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$, for $x \approx x_0$, given in the Data Sheet,

$$(\Omega - \Omega_0) = \left.\frac{d\Omega}{dR}\right|_{R=R_0}(R - R_0)$$ 2.0

$$= \left[\left.\left(\frac{1}{R}\frac{dV}{dR}\right)\right|_{R=R_0} - \left.\left(\frac{1}{R^2}V\right)\right|_{R=R_0}\right](R - R_0)$$

$$= \frac{1}{R_0^2}\left[R_0\left.\frac{dV}{dR}\right|_{R=R_0} - V_0\right](R - R_0)$$

Thus the line-of-sight velocity expression becomes

$$V_r = \frac{1}{R_0^2}\left[R_0\left.\frac{dV}{dR}\right|_{R=R_0} - V_0\right](R - R_0)R_0 \sin l$$

Further, in the solar neighbourhood, $D \ll R_0$. So we can approximate

$$(R - R_0) \approx -D\cos l$$ 1.0

Thus, the line-of-sight velocity can be expressed as,

$$V_r = \frac{1}{R_0^2}\left[R_0\left.\frac{dV}{dR}\right|_{R=R_0} - V_0\right](-D\cos l)R_0 \sin l$$

$$= \frac{1}{2}\left[\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R=R_0}\right] D\sin 2l$$

Comparing with $V_r = AD\sin 2l$, we get

$$\boxed{A = \frac{1}{2}\left[\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R=R_0}\right]}$$ 2.0

Also, in the solar neighbourhood, $\Omega D \approx \Omega_0 D$. Using this approximation and $(R - R_0) \approx -D\cos l$, the tangential velocity is 1.0

$$
\begin{aligned}
V_t &= (\Omega - \Omega_0) R_0 \cos l - \Omega D \\
&= \frac{1}{R_0^2}\left[R_0 \left.\frac{dV}{dR}\right|_{R=R_0} - V_0\right](-D\cos l) R_0 \cos l - \Omega_0 D \\
&= \left[\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R=R_0}\right] D\cos^2 l - \Omega_0 D \\
&= \frac{1}{2}\left[\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R=R_0}\right] D(1+\cos 2l) - \frac{V_0}{R_0} D \\
&= AD\cos 2l + \frac{1}{2}\left[\frac{V_0}{R_0} - \left.\frac{dV}{dR}\right|_{R=R_0}\right] D - \frac{V_0}{R_0} D \\
&= AD\cos 2l - \frac{1}{2}\left[\frac{V_0}{R_0} + \left.\frac{dV}{dR}\right|_{R=R_0}\right] D
\end{aligned}
$$

Comparing with $V_t = AD\cos 2l + BD$, we get

$$
\boxed{B = -\frac{1}{2}\left[\frac{V_0}{R_0} + \left.\frac{dV}{dR}\right|_{R=R_0}\right]}
$$

2.0

Main credit points:

$(\Omega - \Omega_0)$ first order approximation in terms of Taylor series	2.0
$(R - R_0) \approx -D\cos l$ approximation	1.0
$\Omega D \approx \Omega_0 D$ approximation	1.0
Working through to arrive at correct expression for A	2.0
Working through to arrive at correct expression for B	2.0

(T06.3c) The ratio (A/B) of Oort's constants for the two given cases, (I) and (II), are defined as $F_{\rm I}$ and $F_{\rm II}$, respectively. Determine $F_{\rm I}$ and $F_{\rm II}$. **2**

Solution:

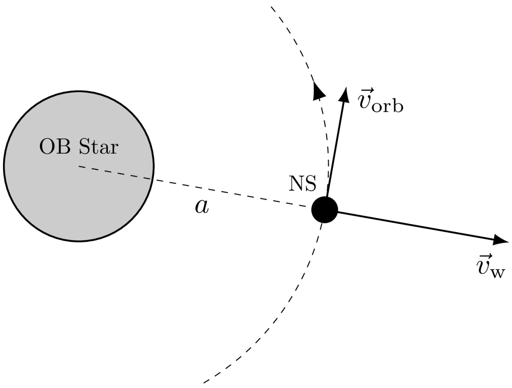
For a flat rotation curve $\left(\frac{dV}{dR} = 0\right)$, $\boxed{F_{\rm I} = -1}$. 1.0

For Keplerian rotation $\left(\frac{dV}{dR} = -\frac{1}{2}\frac{V}{R}\right)$, $\boxed{F_{\rm II} = -3}$. 1.0

(T07) **Neutron Star Binary** **[20 marks]**

In a binary system involving a compact star, where the companion star does not overflow its Roche lobe, the primary source of accretion for the compact star is the stellar wind from the companion star. This wind-fed accretion is especially significant in systems that include an early-type star (such as an O or B star, indicated henceforth as an OB star), alongside a compact object like a neutron star (NS) in a close orbit.

Consider such a NS-OB star binary system where a neutron star of mass $M_{NS} = 2.0\,M_\odot$ and radius $R_{NS} = 11\,\mathrm{km}$ is orbiting in a circular orbit of radius a around the centre of the OB star with velocity $v_{orb} = 1.5 \times 10^5\,\mathrm{m\,s^{-1}}$ (see figure below). Throughout this problem the mass loss from the OB star is assumed to be spherically symmetric and its rate is $\dot{M}_{OB} = 1.0 \times 10^{-4}\,M_\odot\,\mathrm{yr^{-1}}$.



(T07.1) The accretion radius, R_{acc}, is defined as the maximum distance from the NS at which the stellar wind can be captured by the NS. If the stellar wind speed at the orbital distance of the NS is $v_w = 3.0 \times 10^6\,\mathrm{m\,s^{-1}}$, find R_{acc} for the above system in km using standard escape velocity calculations. **3**

Solution:

The accretion radius can be estimated by equating the gravitational potential energy of the stellar wind to its kinetic energy

$$\frac{1}{2}\rho_w v_{rel}^2 = \frac{GM_{NS}\rho_w}{R_{acc}}$$ 0.5

$$\implies R_{acc} = \frac{2GM_{NS}}{v_{rel}^2}$$ 0.5

where, ρ_w is the wind density and v_{rel} is the relative velocity of the wind.

For the case considered above,

$$v_{rel}^2 = v_w^2 + v_{orb}^2$$

Here, $v_w \gg v_{orb}$, therefore, $v_{rel}^2 \approx v_w^2 = 9 \times 10^{12}\,\mathrm{m^2\,s^{-2}}$. 1.0

which gives,

$$R_{acc} = \frac{2 \times 6.674 \times 10^{-11} \times 2.0 \times 1.988 \times 10^{30}}{9 \times 10^{12}}\,\mathrm{m}$$

$$\boxed{R_{acc} = 5.9 \times 10^4\,\mathrm{km}}$$ 1.0

(T07.2) Assuming that all captured material is accreted by the NS, estimate the mass accretion rate, $\dot{M}_{acc}$, from the stellar wind onto the NS in units of $M_\odot\,\mathrm{yr^{-1}}$ if $a = 0.5\,\mathrm{au}$. Neglect the effects of radiation pressure and finite cooling time of the accreting gas. **3**

Solution:
In the case of spherically symmetric mass loss from the companion and $v_\mathrm{w} \gg v_\mathrm{orb}$, the companion mass loss is

$$\dot{M}_\mathrm{OB} = 4\pi a^2 \rho_\mathrm{w} v_\mathrm{w}$$ 1.0

where $a = 0.5\,\mathrm{au} = 7.480 \times 10^{10}\,\mathrm{m}$ is the orbital separation.
Mass accretion rate on the compact object is

$$\dot{M}_\mathrm{acc} \approx \pi R_\mathrm{acc}^2\, \rho_\mathrm{w}\, v_\mathrm{rel}$$ 1.0

Also here, since $v_\mathrm{w} \gg v_\mathrm{orb}$, $v_\mathrm{rel} \approx v_\mathrm{w}$ This implies,

$$\dot{M}_\mathrm{acc} \approx \left(\frac{\dot{M}_\mathrm{OB}}{4}\right)\left(\frac{R_\mathrm{acc}}{a}\right)^2 \approx 9.74 \times 10^{11}\,\mathrm{kg\,s^{-1}}$$

$$\implies \boxed{\dot{M}_\mathrm{acc} \approx 1.6 \times 10^{-11}\,\mathrm{M_\odot\,yr^{-1}}}$$ 1.0

If the student does not use the approximation for speed, then he will get $9.75 \times 10^{11}\,\mathrm{kg\,s^{-1}}$, but the final answer is same for required significant digits. No marks should be deducted if this is the case.

(T07.3) Now consider the situation where the stellar wind speed at the orbital distance a (near the NS) becomes comparable with orbital speed of the NS. The mass accretion rate from the stellar wind onto the NS in this case would be given by an expression of the form $\dot{M}_\mathrm{acc} = \dot{M}_\mathrm{OB} f(\tan\beta, q)$, where $q = M_\mathrm{NS}/M_\mathrm{OB}$ is the mass ratio of the binary and β is the angle in the frame of the NS between the wind velocity direction and radial direction away from the OB star. Obtain the expression for $f(\tan\beta, q)$ assuming $M_\mathrm{OB} \gg M_\mathrm{NS}$. **6**

Solution:
The orbital speed of the NS in the frame corresponding to the center of OB star (assuming the star is stationary) is given by

$$v_\mathrm{orb}^2 = \frac{GM_\mathrm{OB}}{a}$$ 1.0

Using vector algebra we get $\tan\beta = v_\mathrm{orb}/v_\mathrm{w}$. 1.0
Now, the ratio of the mass accretion rate to that of mass loss rate from the star

$$\frac{\dot{M}_\mathrm{acc}}{\dot{M}_\mathrm{OB}} \sim \frac{\pi R_\mathrm{acc}^2 \rho_\mathrm{w} v_\mathrm{rel}}{4\pi a^2 \rho_\mathrm{w} v_\mathrm{w}}$$

$$= \frac{1}{4}\left(\frac{R_\mathrm{acc}}{a}\right)^2 \sqrt{\frac{v_\mathrm{orb}^2 + v_\mathrm{w}^2}{v_\mathrm{w}^2}} = \frac{1}{4}\left(\frac{R_\mathrm{acc}}{a}\right)^2 \sqrt{1 + \tan^2\beta}$$ 1.0

To find the expression of the ratio of the two radii, R_acc and orbital separation a,

$$\frac{R_\mathrm{acc}}{a} = \frac{2GM_\mathrm{NS}}{v_\mathrm{rel}^2}\frac{1}{a} = \frac{2GM_\mathrm{OB}\,q}{v_\mathrm{rel}^2}\frac{1}{a} = \frac{2v_\mathrm{orb}^2}{v_\mathrm{rel}^2} q$$ 1.0

$$= \left(\frac{2\tan^2\beta}{1 + \tan^2\beta}\right) q$$ 1.0

On substituting and simplifying, we can express,

$$\implies \boxed{f(\tan\beta, q) = q^2 \left(\frac{\tan^4\beta}{(1 + \tan^2\beta)^{3/2}}\right)}$$ 1.0

(T07.4) Consider that the fully ionized material accretes radially and is hindered by the strong magnetic field $\vec{B}$ of the NS. The pressure exerted by the magnetic field is given by $\frac{B^2}{2\mu_0}$. We shall assume that the NS has a dipole magnetic field whose magnitude in the equatorial plane varies with the distance r from the NS for $r \gg R_{\rm NS}$ as

$$B(r) = B_0 \left(\frac{R_{\rm NS}}{r}\right)^3$$

where B_0 is the magnetic field at the equator of the NS. Assume that the axis of the magnetic dipole aligns with the rotation axis of the NS.

(T07.4a) Obtain the magnetic pressure, $P_{\rm eq,\,mag}$, in the equatorial plane in terms of B_0, $R_{\rm NS}$, r, and other suitable constants. **1**

Solution:
The magnetic pressure $P_{\rm mag} = B^2/2\mu_0$ increases rapidly towards the NS magnetic poles.

$$\implies \boxed{P_{\rm eq,\,mag} = \frac{B_0^2 R_{\rm NS}^6}{2\mu_0 r^6}}$$ 1.0

(T07.4b) The maximum distance where the accretion flow is stopped by the magnetic field at the equatorial plane is called the magnetospheric radius $R_{\rm m}$. This flow of matter will exert a pressure due to the relative motion between incoming stellar wind and the NS. Obtain an approximate expression for the critical magnetic field $B_{0,\,\rm c}$ for which $R_{\rm m}$ coincides with $R_{\rm acc}$ and calculate its value in Tesla. Magnetic effects are neglected for $r > R_{\rm m}$ and consider $v_{\rm w} \gg v_{\rm orb}$. **7**

Solution:
The critical magnetic field $B_{0,\,\rm c}$ can be obtained by equating pressure due to incoming wind with the $P_{\rm eq,\,mag}$ at $r = R_{\rm acc}$. 1.0
The pressure due to incoming wind at $r = R_{\rm acc}$ is given by

$$\rho_{\rm w} v_{\rm rel}^2 \sim \rho_{\rm w} v_{\rm w}^2$$ 1.0

If the student writes the pressure due to incoming wind as dynamic pressure, i.e., $\frac{1}{2}\rho_{\rm w} v_{\rm rel}^2 \sim \frac{1}{2}\rho_{\rm w} v_{\rm w}^2$, 0.5 credit will be given.

From part (T07.2), one can obtain the expression for $\rho_{\rm w}$ as follows:

$$\rho_{\rm w} = \frac{\dot{M}_{\rm acc}}{\pi R_{\rm acc}^2 v_{\rm w}} = \frac{\dot{M}_{\rm OB}}{4\pi a^2 v_{\rm w}}$$ 1.0

If the student writes the density of wind as $\frac{\dot{M}_{\rm acc}}{4\pi R_{\rm acc}^2 v_{\rm w}}$, 0.5 credit will be given.

Now equating the pressures due to wind and magnetic field, we have:

$$\frac{B_{0,\,\rm c}^2 R_{\rm NS}^6}{2\mu_0 R_{\rm acc}^6} = \frac{\dot{M}_{\rm acc}}{\pi R_{\rm acc}^2}\left(\frac{2GM_{\rm NS}}{R_{\rm acc}}\right)^{1/2}$$ 1.0

which gives the value of the critical magnetic field as

$$\implies \boxed{B_{0,\,\rm c} = \left(\frac{2\mu_0 \dot{M}_{\rm acc}\sqrt{2GM_{\rm NS}}}{\pi R_{\rm NS}^6}\right)^{1/2} R_{\rm acc}^{7/4}}$$ 1.0

No credit will be given if the expression has wrong exponent, or is missing a constant quantity.

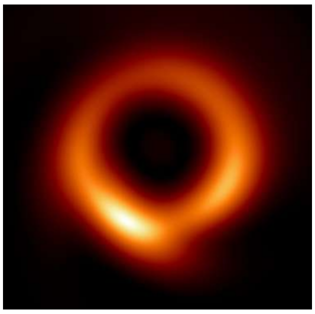
Now putting the values we get

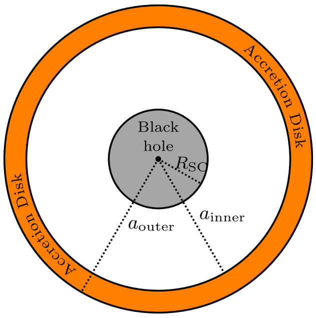
$$\implies \boxed{B_{0,\,\mathrm{c}} \approx 4.0 \times 10^{9}\ \mathrm{T}}$$ 2.0

(T08) Shadow of a black hole [20 marks]

The Event Horizon Telescope (EHT) has released an image of the supermassive black hole at the centre of the M87 galaxy as shown in the left panel of the figure below.

To understand some simple features of this image, we will consider a simplified model of a non-rotating, static, spherically symmetric black hole of mass $M = 6.5 \times 10^9\,\mathrm{M}_\odot$, surrounded by a massless, thin, planar accretion disk of inner and outer radii, $a_{\text{inner}} = 6R_{\text{SC}}$ and $a_{\text{outer}} = 10R_{\text{SC}}$, respectively, where R_{SC} is the Schwarzschild radius. A face-on view sketch is shown in the right panel of the figure below (figure is not to scale).





We assume that the accretion disk is the only source of light to be considered. Every point on the disk emits light in all directions. This light travels under the influence of the gravitational field of the black hole. The path of the light rays is governed by two equations given below (which are similar to those of an object around the Sun):

$$\frac{1}{2}v_{\text{r}}^2 + \frac{L^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) = E \quad ; \qquad v_\phi = r\,\omega = \frac{L}{r}$$

where $r \in (R_{\text{SC}}, \infty)$ is the radial coordinate, $\phi \in [0, 2\pi)$ is the azimuthal angle, and E and L are constants related to the conserved energy and conserved angular momentum, respectively.

Here $v_{\text{r}} \equiv dr/dt$ is the magnitude of the radial velocity, v_ϕ is the magnitude of the tangential velocity, and $\omega \equiv d\phi/dt$ is the angular velocity. We define the impact parameter b for a trajectory as $b = L/\sqrt{2E}$. Time dilation is neglected in this problem.

Another useful equation is obtained by differentiating the first equation:

$$\frac{dv_{\text{r}}}{dt} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2 r^4} = 0$$

(T08.1) Circular light trajectories can exist around the black hole. Find the radius, r_{ph}, and impact parameter, b_{ph}, for such photon trajectories in terms of M and relevant constants. **4**

Solution:

Circular trajectories have $r =$ constant. Hence, the radial velocity $v_{\text{r}} = 0$ 1.0

and rate of change of radial component of velocity $\dfrac{dv_{\text{r}}}{dt} = 0$. 1.0

From the third equation,

$$\frac{dv_{\text{r}}}{dt} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2 r^4} = 0,$$

we get radius r_{ph} of circular orbits to be $\boxed{r_{\text{ph}} = 3GM/c^2}$ 1.0

Then from the first equation, the impact parameter is found to be

$\boxed{b_{\text{ph}} = L/\sqrt{2E} = 3\sqrt{3}GM/c^2}$. 1.0

(T08.2) Calculate the time T_{ph} taken for completing one full orbit of the circular light trajectory in seconds. **2**

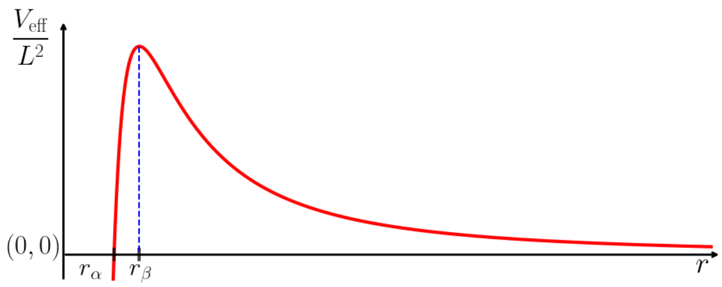
Solution:
Time taken to complete one circular orbit is

$$T_{\mathrm{ph}} = 2\pi r_{\mathrm{ph}}/c = 6\pi GM/c^3 \quad 1.0$$

$$= \boxed{6.0 \times 10^5\ \mathrm{s}} \quad 1.0$$

Half credit lower limit	Full credit range	Half credit upper limit
	5.9×10^5 s to 6.1×10^5 s	

(T08.3) The radial velocity equation given above (the first equation in this question) for light trajectories can be compared with an equation of the form $\frac{v_r^2}{2} + V_{\mathrm{eff}}(r) = E$. A schematic plot of V_{eff}/L^2 as a function of r is given below.



(T08.3a) The plot indicates two special radii, r_α and r_β. Obtain expressions for r_α and r_β in terms of M and relevant constants. **2**

Solution:
r_α corresponds to the radius where $V_{\mathrm{eff}} = \frac{L^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right)$ is equal to zero.

Hence $\boxed{r_\alpha = 2GM/c^2} = R_{\mathrm{SC}}$. 1.0

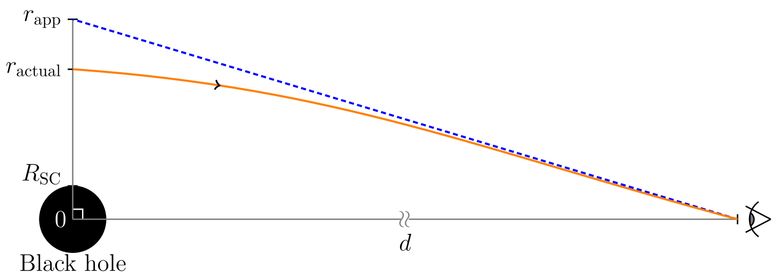
Solving for $\frac{dV_{\mathrm{eff}}}{dr} = 0$ gives $\boxed{r_\beta = 3GM/c^2} = r_{\mathrm{ph}}$. 1.0

(T08.3b) A photon travelling inward from the accretion disk towards the black hole can still escape out to infinity in some cases. Find the expression for the smallest value of the turning point radius, r_t, for such a photon, in terms of M and relevant constants. Find the expression for the minimum value of the impact parameter, b_{min}, for this photon. **3**

Solution:
The radius of the position of peak of $V_{\mathrm{eff}}(r)$ is the smallest possible turning point radius r_t for the light trajectory such that it can escape to infinity. 1.0
Hence, the least smallest possible turning point radius r_t is same as the radius r_{ph} of the circular orbits. $\boxed{r_t = 3GM/c^2}$ 1.0

Since r_t is equal to r_{ph}, the corresponding impact parameter is same as calculated in part one $\boxed{b_{\mathrm{min}} = 3\sqrt{3}GM/c^2}$. 1.0

(T08.4) A ray of light coming from a radius r_{actual} from the centre of the system in the plane of the sky will suffer strong bending due to the gravity of the black hole, and eventually **5** reach an observer (denoted by an eye) at a large distance d from the system, as shown below.



To this observer, the ray would appear to have originated from a different point at a distance $r_{app} \approx b$ from the black hole centre in the plane of the sky, where b is the impact parameter for that photon trajectory. For points on the accretion disk at $r = r_{actual}$, one may assume the following relation:

$$b(r_{actual}) \approx r_{actual} \left(1 + R_{SC}/r_{actual}\right)^{1/2}$$

For the distant observer, like ourselves, with a face-on view of the accretion disk, the image of the system will appear to be circularly symmetric in the plane of the sky. Determine the outermost apparent radius, r_{outer}, and the innermost apparent radius, r_{inner}, of the image in units of au.

Solution:

Since it is given that $r_{app} \approx b$, the apparent radii r_{inner} and r_{outer} are related to the respective impact parameters b_{inner} and b_{outer} of the corresponding light rays.

To figure out the b_{inner} and b_{outer}, one needs to choose the correct corresponding r_{actual} in each case.

The outer ring radius r_{outer} would correspond to the light ray which is emitted from the outer-edge of the accretion disk. Hence the r_{outer} is determined by the formula given $r_{outer} \approx b_{outer} \approx a_{outer} \left(1 + R_{SC}/a_{outer}\right)^{1/2}$. 1.0

Substituting $a_{outer} = 10R_{SC}$ we get $r_{outer} = 1.3 \times 10^3$ au 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	1.2×10^3 au to 1.4×10^3 au	

For the inner ring radius r_{inner}, the light ray corresponding to the smallest turning point r_t is the light ray which goes closest to the BH and returns back to infinity.

Thus one should choose the impact parameter b_{inner} to be that of the circular orbit.

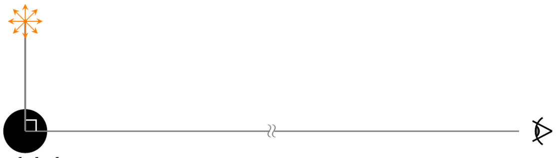
Then, $r_{inner} \approx b_{inner} = b_{ph} = 3\sqrt{3}GM/c^2$. 2.0

Hence $r_{inner} = 3.3 \times 10^2$ au . 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	3.2×10^2 au to 3.4×10^2 au	

If a student calculates the r_{inner} corresponding to a_{inner}, then only 1.0 mark out of 3.0 marks will be awarded provided the final number is within the range $r_{inner} = (8.3 \pm 0.1) \times 10^2$ au.

(T08.5) Consider an isolated supermassive black hole of mass $M = 6.5 \times 10^9\,M_\odot$ without any accretion disk. A brief strong burst of electromagnetic radiation occurs for 5 s at a point Z at a distance, say, $r_Z = 6R_{SC}$ from the black hole as shown in the figure. The burst at point Z emits light in all directions. An observer at a point far from the black hole (denoted by an eye in the figure below) takes a long exposure image of the region around the black hole for 60 s.



Choose the correct option for each of the statements below:

(T08.5a) The number of possible paths for light to travel from Z to the observer is **2**
(A) At most one (B) Exactly one (C) Exactly two (D) Greater than two.

Solution:
The source is a point source.
For the image, one has to look at all the possible trajectories light that can reach the observer starting from point Z in the same plane as the observer, the black hole and point Z.
The shortest path is the direct path, without orbiting the black hole, connecting Z and the observer. This will be also be bent due to gravity. The next shortest path could be the one which goes half around the black hole in the anti-clockwise sense in the picture. Another one is the one with one and half revolution around the black hole in the anti-clockwise sense in the picture. And so on, paths with additional one more revolution around the black hole than the previous one.
The above is also true for trajectories in clockwise sense of rotation around the black hole. Therefore, number of possible paths for light to travel from Z to the observer is (D) Greater than two 2.0

(T08.5b) The number of images of the EM burst at Z that will be seen in the long exposure image is **2**
(A) At most one (B) Exactly one (C) Exactly two (D) Greater than two.

Solution:
One has to consider the time difference in the arrival times due to the different path lengths. From Part (T08.2), it takes around 6.0×10^5 sec which is 166.7 h for light to circularly orbit the supermassive black hole of mass $M = 6.5 \times 10^9\,\mathrm{M}_\odot$. Hence, the difference in the arrival times of light along these trajectories would be at least around half of 166.7 h approximately, that is, around 83.3 h. Each of these signals would last for only the duration of the blast, i.e., 5 s.
Since the long exposure picture is only for 60 s, either one or no image will be captured, depending on whether the exposure time encompasses the short window of the arrival of one of the signals and 5 s thereafter. Therefore, the number of images in a picture would be (A) At most one . 2.0

A correct answer in this question will fetch 2.0 marks if either option (C) or option (D) were marked in 8.5a. If the answer in 8.5a is either of (A) or (B) or is not attempted, then a correct answer in this question fetches 1.0 mark.

(T09) **Atmospheric Seeing** **[35 marks]**

A telescope with an achromatic convex objective lens of diameter $D = 15\,\text{cm}$ and focal length $f = 200\,\text{cm}$ is pointed to a star at the zenith.

(T09.1) Find the diameter (in m), d_{image}, of the image of a point source as produced by the objective lens at its focal plane for green light ($\lambda = 550\,\text{nm}$), considering only the effects of diffraction. **1**

Solution:

Angular radius of the diffraction disk is given by $\dfrac{1.22\lambda}{D}$.

The radius of the image is therefore,

$$d_{\text{image}} = f\left(\frac{1.22\lambda}{D}\right)$$ 0.5

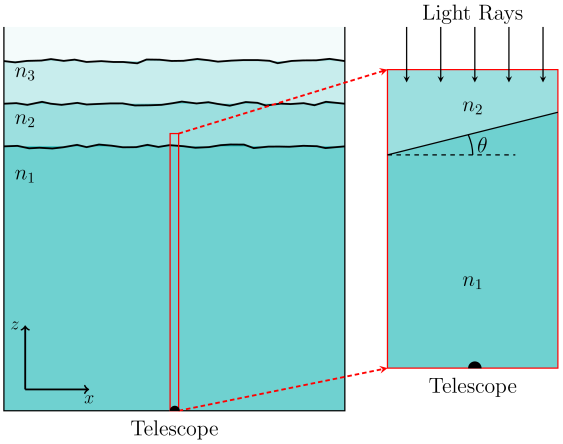
$$\boxed{d_{\text{image}} = 8.95 \times 10^{-6}\,\text{m}}$$ 0.5

We accept $f\left(\dfrac{\lambda}{D}\right) = 7.33 \times 10^{-6}\,\text{m}$ (i.e., without the factor of 1.22 in the formula). No other values accepted.

The image of an astronomical source is also affected by the so-called "atmospheric seeing".

The boundaries between the layers in the atmosphere as well as the refractive indices of the layers change continuously due to turbulence, temperature variation and other factors. This leads to tiny changes in the position of the image in the focal plane of the telescope, known as the "twinkling effect". For rest of the problem, apart from the diffraction limited finite size of the image of the star discussed above, no interference effects will be considered.

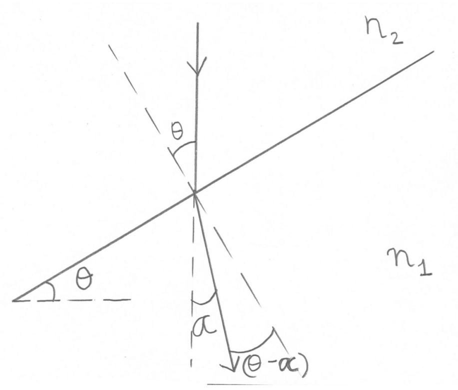
The left panel of the figure below shows a vertical cross-section of the atmosphere with multiple layers of different refractive indices ($n_1, n_2, n_3, \ldots$). The right panel shows the zoomed in view of a thin vertical segment of the atmosphere and the boundary between the two lowest atmospheric layers of refractive indices n_1 and n_2 ($n_1 > n_2$). We consider only these two layers and their boundary for this problem. The diagrams are not to scale.



(T09.2) Let the boundary between the two layers be at a height $H = 1\,\text{km}$ directly above the telescope objective, with a tilt of $\theta = 30^\circ$ with respect to the horizontal plane. In all parts of this problem θ is taken to be positive in the anti-clockwise direction. For a monochromatic light source, $n_1 = 1.00027$ and $n_2 = 1.00026$. Let the angular shift in the position of the image at the focal plane of the telescope for a star at the zenith be α.

(T09.2a) Draw an appropriately labelled ray-diagram at the boundary showing n_1, n_2, θ and α. **2**

Solution:



2.0

Incident, refracted rays with arrows and normal to boundary drawn	0.5
Angle of incidence shown as θ	0.5
Angle of shift in refracted ray shown as α	0.5
Both n_1 and n_2 shown	0.5

(T09.2b) Find the expression for α in terms of θ, n_1 and n_2. Use the small angle approximations: $\sin\alpha \approx \alpha$ and $\cos\alpha \approx 1$. **2**

Solution:
Using Snell's Law,

$$n_2 \sin\theta = n_1 \sin(\theta - \alpha)$$

$$\frac{n_2}{n_1}\sin\theta = \sin\theta\cos\alpha - \cos\theta\sin\alpha$$ 1.0

Using small angle approximation ($\sin\alpha \approx \alpha$ and $\cos\alpha \approx 1$)

$$\frac{n_2}{n_1}\sin\theta = \sin\theta - \alpha\cos\theta$$

$$\boxed{\alpha = \frac{\sin\theta}{\cos\theta}\left(1 - \frac{n_2}{n_1}\right)}$$ 1.0

(T09.2c) Calculate the displacement, Δx_θ (in m), in the position of the image if θ increases by 1% (keeping n_1 and n_2 fixed). **3**

Solution:
The shift in position of image due to change in θ is:

$$\Delta x_\theta = f\alpha = f\Delta\tan\theta\left(1 - \frac{n_2}{n_1}\right)$$ 1.0

$$= f\left(1 - \frac{n_2}{n_1}\right)(\tan(\theta + \Delta\theta) - \tan\theta)$$

$$= f\left(1 - \frac{n_2}{n_1}\right)\sec^2\theta\Delta\theta$$ 1.0

$$= f \times (6.98 \times 10^{-8}\,\text{rad})$$

$$\boxed{\Delta x_\theta = 1.40 \times 10^{-7}\,\text{m}}$$ 1.0

Wrong sign will have penalty of 1.0 mark.

(T09.2d) Calculate the displacement, Δx_n (in m), in the position of the image if n_2 increases by 0.0001% (keeping n_1 and θ fixed). **3**

Solution:
Shift in the position of image due to the change in n_2 is:

$$\Delta x_n = f\tan\theta\left(-\frac{n_2+\Delta n_2}{n_1}+\frac{n_2}{n_1}\right)$$

$$= -f\tan\theta\left(\frac{0.0001}{100}\times\frac{n_2}{n_1}\right) \quad 2.0$$

$$= -f\times(5.77\times10^{-7}\,\mathrm{rad})$$

$$\boxed{\Delta x_n = -1.15\times10^{-6}\,\mathrm{m}} \quad 1.0$$

Wrong sign will have penalty of 1.0 mark.

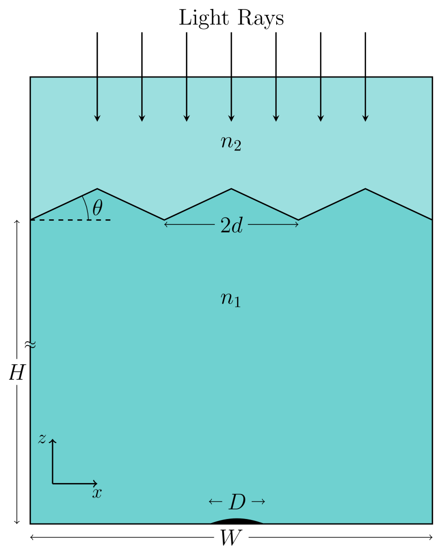
(T09.3) For white light coming from a star at the zenith, choose which of the following most closely describes the shape and colour of the image by ticking (✓) the appropriate box (only one) in the Summary Answersheet. Note x increases from left to right in the figure. **2**

	Image colour	Image shape	Left edge	Right edge
A	White	Circular		
B	White	Elliptical		
C	Coloured	Circular	Blue	Red
D	Coloured	Circular	Red	Blue
E	Coloured	Elliptical	Blue	Red
F	Coloured	Elliptical	Red	Blue

Solution:
Blue light has larger refractive index than the red light, hence it will bend more. So incoming light will vary in colour from red to blue towards right.
So, the correct answer will be F . 2.0
Note: convex lens inverts image, but does not affect refractive index dependent bending.
Options D or E get 1.0 mark.

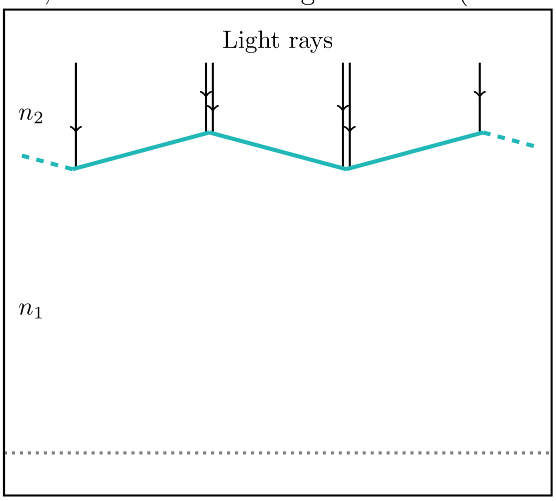
For all remaining parts of this question we consider monochromatic green light with $\lambda = 550\,\text{nm}$. We model the boundary between the layers as a set of infinite zigzag planes (running perpendicular to the plane of the page) separated by $d = 10\,\text{cm}$ along x-axis, with either $\theta = 10°$ or $\theta = -10°$.

The figure below (not to scale) shows a cross-section of this model of the atmosphere of width W ($W \ll H$). For telescopes with large aperture this zigzag nature of the boundary results in formation of speckles in the focal plane.



(T09.4) Consider an atmosphere modelled as above.

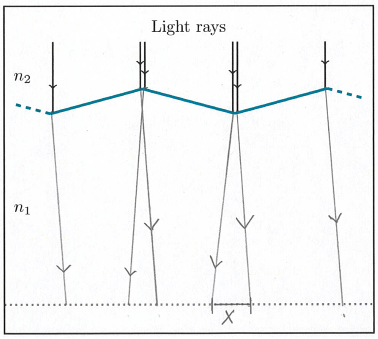
(T09.4a) A section of the atmosphere with consecutive zigzag planes, with same parameters as stated above, is shown in the diagram below (not to scale). **4**



In this diagram, reproduced in the Summary Answersheet, draw the paths of the incident light rays up to the plane where the telescope objective is placed, shown by the gray dotted line.

Mark the region(s), if any, by "X" in the diagram where no light rays will reach.

Solution:



4.0

All three sets of parallel rays drawn with correct inclinations	2.0
All rays have arrow marks	0.5
X-patch marked correctly	1.5

(T09.4b) Calculate the width W_X of such region(s). **3**

Solution:
Each set of planes, with specific tilt, will lead to one set of stripes of parallel rays. From solution of part (T09.2b),

$$\alpha = \tan\theta \frac{(n_1 - n_2)}{n_1}$$

Two sets of planes with relative angle of 2θ will create angular separation of 2α between the stripes. 1.0
At a distance of H this will create a horizontal separation of $\approx 2\alpha H$.
For $H = 1\,\text{km}$, $\theta = 10°$, the horizontal shift will be

$$\Delta x \simeq 2\alpha H = 2\tan\theta \frac{(n_1 - n_2)}{n_1} H = 2 \times 1.76 \times 10^{-6} \times 1 \times 10^{5}\,\text{cm} = 0.35\,\text{cm}.$$ 1.0

Just below the point where two planes make upward bend, a patch of this width 0.35 cm will be centred where no light from the star will reach.
Hence, $\boxed{W_X = 0.35\,\text{cm}}$ 1.0

(T09.4c) Find the largest diameter, D_{max}, of the telescope objective with which it will be possible to obtain a single image of a star, by appropriately choosing the location of the telescope relative to the structure of the boundary. **4**

Solution:
From the figure one can see that two adjacent planes (with downward bend), will lead to overlapping stripes of rays producing two images.
Therefore, the region in which only single image is formed will be 10.0 cm (including the region X).
Hence, $\boxed{D_{max} = 10.0\,\text{cm}}$ 4.0

$(10\,\text{cm} - 2W_X)$ will get 1.0 mark
$(10\,\text{cm} - W_X)$ will get 2.0 marks.

(T09.5) Consider the case when the zigzag shape of the boundary is allowed in both x and y directions, (like a field of pyramids), and $D = 100\,\text{cm}$ (with $f = 200\,\text{cm}$). **6**

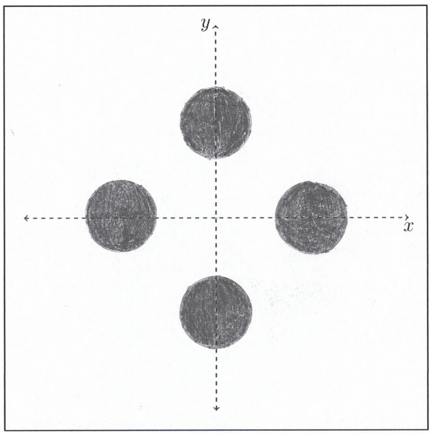
Draw the qualitative pattern of the resulting speckles in the box given in the Summary Answersheet.

Solution:
For $D \gg d$, several stripes will fall on the lens, leading to multiple images. Here, four images will form, two each from the set of planes with zigzag shapes along x-axis and y-axis respectively.

First consider the zigzag shape of the boundary along the x axis. From solution of (T09.4), angle between these two sets of rays will be 3.52×10^{-6} rad. With $f = 2$ m, the separation between the two images will be $= 7.0\,\mu$m.

For $D = 100$ cm, the diffraction peak size $= 1.34\,\mu$m. So the images formed by the two sets of planes (with tilts of $\pm 10°$ from horizontal for each set) will form two spots, clearly separated along the x-axis. Similarly for the y-axis.

The resulting speckles will be as shown below.



4 spots drawn	3.0
Spots along x and y axes	1.0
Circular shape of spots	1.0
No overlap of spots	1.0

(T09.6) For a turbulent atmosphere again consider the same parallelly running zigzag shape of the boundary layer only along x-direction, but now the angle between two planes are changing at a uniform rate from $10°$ to $-10°$ in 1.0 s. Assume that this leads to a uniform rate of shift of the position of the image. **5**
Consider a telescope with $D = 8$ cm and $f = 1$ m. Estimate the longest exposure time $t_{\max}$ allowed for its CCD camera so that one gets only a single image, and any possible deviation in its position remains less than 1 % of the diffraction limited diameter of the image.

Solution:
A single image is formed when only one stripe of rays is allowed to fall on the objective. This is only possible when the objective is placed suitably below the zigzag

pattern.

The diffraction-limited image size for a $D = 8\,\text{cm}$ lens is $8.4 \times 10^{-6}\,\text{m}$. 1.0

The angle of the atmospheric layer changes from $+10°$ to $-10°$
$\implies$ Change in angle of rays is 3.52×10^{-6} rad
$\implies$ Shift in image position $= 3.52\,\mu\text{m}$. 1.0

Now, shift in the image position allowed is $1\,\%$ of the image size.
$\therefore$ Allowed shift $= 0.084\,\mu\text{m}$ 1.0

Longest exposure time = Time taken by the image to shift by $0.084\,\mu\text{m}$

Image shifts by $3.52\,\mu\text{m}$ in $1\,\text{s}$ $\implies$ $0.084\,\mu\text{m}$ shift occurs in $0.024\,\text{s}$.
$t_{\text{max}} = 0.024\,\text{s}$ 2.0

(T10) **Big Bang Nucleosynthesis** **[35 marks]**

During the radiation dominated era in the early Universe, the scale factor of the Universe $a \propto t^{1/2}$, where t is the time since Big Bang. During most of this era, neutrons (n) and protons (p) remain in thermal equilibrium with each other via weak interactions. The number density (N) of free neutrons or protons is related to the temperature T and their corresponding masses m such that

$$N \propto m^{3/2} \exp\left(-\frac{mc^2}{k_\mathrm{B}T}\right),$$

as long as time $t \leq t_\mathrm{wk} = 1.70\,\mathrm{s}$, when $k_\mathrm{B}T \geq k_\mathrm{B}T_\mathrm{wk} = 800\,\mathrm{keV}$. After t_wk, the weak interactions can no longer maintain such equilibrium, and free neutrons decay to protons with a half-life time of 610.4 s.

(T10.1) Let the number density of protons be N_p, and that of neutrons be N_n. Calculate the relative abundance of neutrons given the ratio $X_\mathrm{n,\,wk} = N_\mathrm{n}/(N_\mathrm{n} + N_\mathrm{p})$ at time t_wk. **4**

Solution:
The ratio of the number density of protons to that of neutrons at t_wk will be given by

$$\frac{N_\mathrm{n}(t_\mathrm{wk})}{N_\mathrm{p}(t_\mathrm{wk})} = \left(\frac{m_\mathrm{n}}{m_\mathrm{p}}\right)^{3/2} \exp\left(-\frac{(m_\mathrm{n} - m_\mathrm{p})c^2}{k_\mathrm{B}T_\mathrm{wk}}\right)$$ 2.0

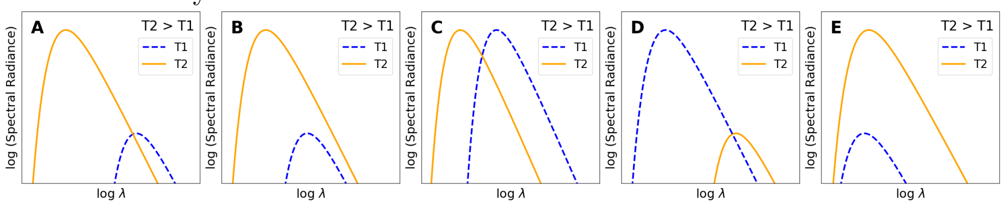
Substituting the values of the masses of neutron and proton, and $k_\mathrm{B}T_\mathrm{wk} = 800\,\mathrm{keV}$, one obtains $X_\mathrm{n}(t_\mathrm{wk}) = \frac{N_\mathrm{n}}{N_\mathrm{p}+N_\mathrm{n}} = 0.166$ 2.0

(T10.2) Photons maintain thermal equilibrium and retain a blackbody spectrum at all epochs.

(T10.2a) Find the index β, such that $T(a) \propto a^\beta$. **2**

Solution:
For a black body spectrum, $\lambda_\mathrm{max}T$ should be equal to a constant. 1.0
Since the wavelength of light is $\lambda \propto a$, this implies $T \propto a^{-1}$, which implies $\beta = -1$. 1.0
Alternatively, for black body radiation, energy density is proportional to T^4, but the energy density is proportional to a^{-4}. Therefore, $T \propto a^{-1}$.

(T10.2b) Identify which of the following graphs shows the correct behaviour of the spectral energy density for two temperatures T_1 and T_2. Tick (✓) the correct option in the Summary Answersheet. **2**



Solution:
Graph B should be ticked. Wien's law removes options D and E, Planck curves at different temperatures should not intersect, and so C and A are also incorrect.

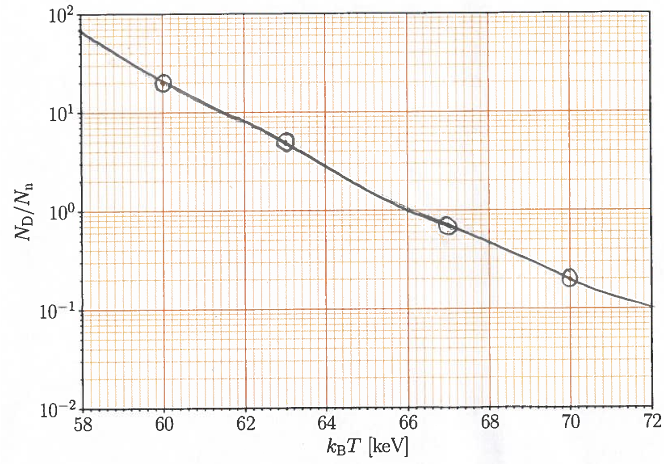
(T10.3) After t_wk, the process of formation of deuterium from protons and neutrons is governed by the Saha equation, given by the Indian physicist Prof. Meghnad Saha, which can be simplified to

$$\frac{N_{\rm D}}{N_{\rm n}} = 6.5\eta \left(\frac{k_{\rm B}T}{m_{\rm n}c^2}\right)^{3/2} \exp\left(-\frac{(m_{\rm D}-m_{\rm p}-m_{\rm n})c^2}{k_{\rm B}T}\right) .$$

Here, baryon-to-photon ratio η is 6.1×10^{-10}, and $N_{\rm D}$ is the number density of deuterium.

(T10.3a) Plot the ratio $N_{\rm D}/N_{\rm n}$ on the grid in the Summary Answersheet, for at least 4 reasonably spaced values of temperature that lie in the domain $k_{\rm B}T = [60, 70]$ keV, and draw a smooth curve passing through these points. **5**

Solution:



Each of the 4 points correctly located on the graph	1.0
Drawing a smooth curve passing through all the points (within one grid point)	1.0

Non-smooth lines, or straight lines not passing through all points get 0 credit.

(T10.3b) From the plot find $k_{\rm B}T_{\rm nuc}$ (in keV) when $N_{\rm D} = N_{\rm n}$. **1**

Solution:
The value of $k_{\rm B}T_{\rm nuc} = 66.0\,{\rm keV}$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	65.4 keV to 66.6 keV	

(T10.3c) Instead, now assume that all the free neutrons combine instantaneously with the protons at $k_{\rm B}T_{\rm nuc}$ to form Deuterium, and all of which immediately gets converted to Helium ($^4_2$He). Compute the corresponding epoch or time of nucleosynthesis, $t_{\rm nuc}$ (in s), for the formation of Helium. **4**

Solution:
As $T \propto a^{-1}$, and $a \propto t^{1/2}$, we have $T \propto t^{-1/2}$. 2.0
The temperature and time when all neutrons and protons fuse into Helium are $T_{\rm nuc}$ (66.0 keV) and $t_{\rm nuc}$, respectively.

$$t_{\rm nuc} = t_{\rm wk}\left(\frac{T_{\rm wk}}{T_{\rm nuc}}\right)^2 = 250\,{\rm s}$$ 2.0

(T10.4) Calculate the value of $X_{\rm n,\,nuc}$ immediately before $t_{\rm nuc}$. **5**

Solution:

During $t_{\rm wk} < t < t_{\rm nuc}$, as the Universe cools from 800 keV to 66.0 keV, neutrons decay into protons and with a half life of 610.4 s, which corresponds to a mean life time $\tau = 880.6$s. 1.0

The quantity $(N_{\rm p} + N_{\rm n})a^3$ does not change as the increase in protons is compensated by the decrease in neutrons. 1.0

However, $N_{\rm n}a^3$ will go down by an exponential factor compared to its value computed in the first question as the neutrons decay into protons, i. e.,

$$X_{\rm n,\,nuc} = X_{\rm n,\,wk}\exp(-(t_{\rm nuc} - t_{\rm wk}/880.6))$$ 2.0

$$\boxed{X_{\rm n,\,nuc} = X_{\rm n,\,wk}\exp(-(250 - 1.70)/880.6) = 0.125}$$ 1.0

Give part credit of 1 marks if half life time is used instead of the mean life time in the denominator of the exponential.

(T10.5) The primordial Helium abundance, $Y_{\rm prim}$, is defined to be the fraction of total mass in the Universe that is bound in Helium at $T_{\rm nuc}$. Obtain a theoretical estimate for the value of $Y_{\rm prim}$. For the purpose of this calculation alone, assume $m_{\rm p} \approx m_{\rm n}$ and that the mass of Helium, $m_{\rm He} \approx 4m_{\rm n}$. **3**

Solution:

Helium forms with the combination of 2 protons and 2 neutrons (ignoring mass deficit), and thus a number density $N_{\rm n}$ of neutrons and correspondingly same number density of protons will be locked in Helium. The number density of both species goes as a^{-3}, and so will cancel from the numerator and denominator. 1.0

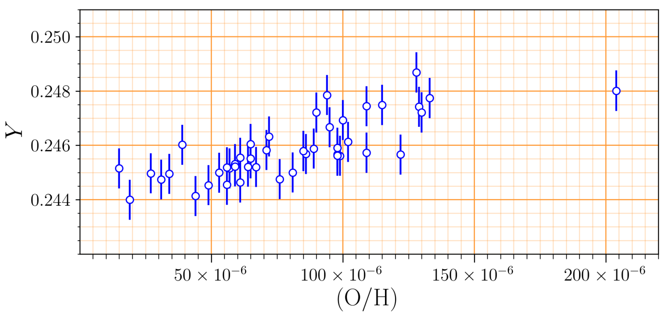
Even after Helium is formed the sum of the mass of Helium and Hydrogen will still be the mass of all the neutrons and protons. 1.0

Thus, the abundance of Helium by mass should be

$$\boxed{Y_{\rm prim} = \frac{(m_{\rm p}+m_{\rm n})N_{\rm n}}{m_{\rm p}N_{\rm p}+m_{\rm n}N_{\rm n}} \approx 2X_{\rm n,\,nuc} = 0.250}$$ 1.0

Alternatively, the value of $X_{\rm n,\,nuc}$ is 1/8. After formation of Helium all neutrons will combine with protons. Therefore, the mass in Helium will be 2/8 of the total mass.

(T10.6) The primordial abundance of Helium is very difficult to measure, as stars continuously convert Hydrogen to Helium in the Universe. The amount of processing by stars in a galaxy is characterised by the relative number density of Oxygen (which is only produced by stars) to hydrogen, denoted as (O/H), in the galaxy. A compilation of the measurements of (O/H) and the Helium abundance, Y, for different galaxies is plotted below.

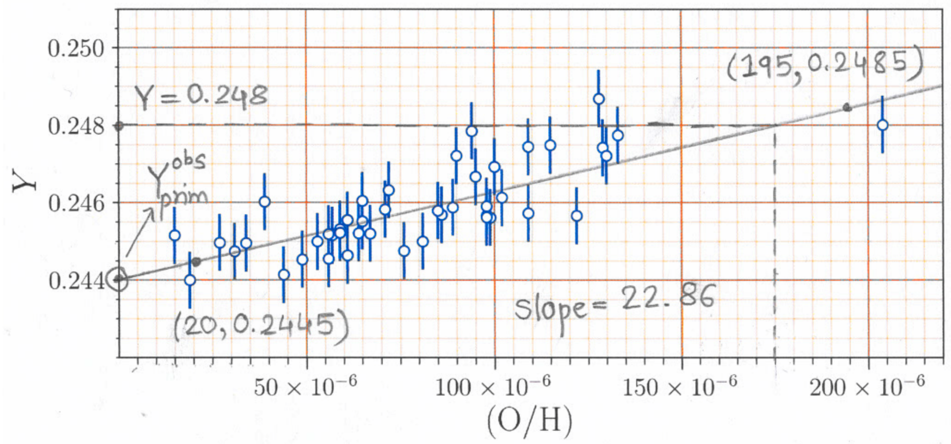


Use all of the points in this plot (which is reproduced in the Summary Answersheet) to answer the following.

(T10.6a) Estimate Y for a blue compact dwarf galaxy with a value of (O/H)=1.75×10^{-4}. **2**

Solution:
By fitting a line to these data points the Helium abundance for the said galaxy would be: $Y = 0.2480$. 2.0

Half credit lower limit	Full credit range	Half credit upper limit
0.2470	0.2475 to 0.2485	0.2490



(T10.6b) Obtain the slope $dY/d(\mathrm{O/H})$ of the straight line fit to the above data. **2**

Solution:
The slope of the line yields $dY/d(\mathrm{O/H}) = 23$. 2.0

Half credit lower limit	Full credit range	Half credit upper limit
15	19 to 27	31

(T10.6c) Estimate the primordial Helium abundance, $Y^{\mathrm{obs}}_{\mathrm{prim}}$, based on the above observations. **2**

Solution:
The primordial Helium abundance, $Y^{\mathrm{obs}}_{\mathrm{prim}} = 0.2440$. 2.0

Half credit lower limit	Full credit range	Half credit upper limit
0.2430	0.2435 to 0.2445	0.2450

(T10.7) The deviation between Y_{prim} and $Y^{\mathrm{obs}}_{\mathrm{prim}}$ can be reconciled by changing the baryon-to-photon ratio η. When η is decreased, as indicated by $\downarrow$ in the Summary Answersheet, indicate the increase ($\uparrow$) or decrease ($\downarrow$) in $N_{\mathrm{D}}/N_{\mathrm{n}}(T)$, T_{nuc} (when $N_{\mathrm{D}} = N_{\mathrm{n}}$), t_{nuc}, $X_{\mathrm{n,\,nuc}}$, and Y_{prim} in the boxes provided in the Summary Answersheet. **3**

Solution:
From the Saha equation, we see that $N_{\mathrm{D}}/N_{\mathrm{n}}(T) \propto \eta$, thus when $\eta \downarrow$, $N_{\mathrm{D}}/N_{\mathrm{n}}(T)$

will also ↓. From the plot in (T10.3a), this would mean that $N_D/N_n(T)$ would be unity at a lower value of T_{nuc} which will go ↓. A lower temperature corresponds to a longer t_{nuc} (↑). This would result in more neutrons decaying into protons and thus less Y_{prim} ↓.

η	↓	$N_D/N_n(T)$	↓	T_{nuc} $(N_D = N_n)$	↓
t_{nuc}	↑	$X_{n,\,nuc}$	↓	Y_{prim}	↓

Each correct box fetches 0.5 marks.

We give a bonus 0.5 marks if all 5 boxes are marked correctly.

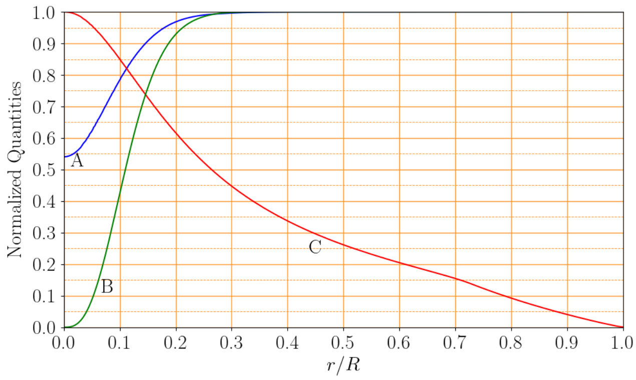
(T11) **Stars through graphs** **[50 marks]**

Stars can be well approximated as spherically symmetric objects, and hence the radial distance r from the centre can be chosen as the only independent variable in modelling stellar interiors. The mass contained within a sphere of radius r is denoted by $m(r)$. The luminosity $l(r)$ is defined as the net energy flowing outward through a spherical surface of radius r per unit time. Other quantities of interest, for example, the density $\rho(r)$, temperature $T(r)$, hydrogen mass fraction $X(r)$, helium mass fraction $Y(r)$, and the nuclear energy generated per unit mass per unit time $\epsilon_{\rm nuc}(r)$, are taken to be functions of r. Throughout this problem we shall neglect the effects of diffusion and gravitational settling of elements inside the star.

The symbol "log" refers to logarithm to the base 10. The problem consists of three independent parts.

(T11.1) **Part 1: Inside a star**

The graph below shows the variation of three structural quantities, A, B, and C, as functions of the fractional radius r/R in a stellar model of mass $1\,\mathrm{M}_\odot$ and age 4 Gyr, where R is the photospheric radius of the star. The values of the helium mass fraction at the (photospheric) surface, $Y_{\rm s}$, and the metallicity (mass fraction of all elements heavier than helium) at the (photospheric) surface, $Z_{\rm s}$, of the star are given by $(Y_{\rm s}, Z_{\rm s}) = (0.28, 0.02)$. All quantities shown in the plots are normalised by their respective maximum values.



(T11.1a) Identify the three quantities A, B, and C uniquely from among the five possibilities: **6**

$$T(r),\ l(r),\ \epsilon_{\rm nuc}(r),\ X(r),\ Y(r).$$

(Write A/B/C in the boxes beside the appropriate quantities in the Summary Answersheet. No justification is needed for your answer.)

Solution:

- X and l increase with r, while T, $\epsilon_{\rm nuc}$, and Y decrease with r. So A and B must be X and l.
- For a $1\,\mathrm{M}_\odot$ star of age 4 Gyr, $X \neq 0$ at centre. So B cannot be X.
- Also, $l = 0$ at centre. Matches with B.
- Both X and l must saturate to their maximum values just outside the H-burning core, and this happens at $r/R = 0.3$ here.
- Each of T, $\epsilon_{\rm nuc}$, and Y have their maximum value at the centre and decrease outwards. So C is one of these.

- But Y cannot be (close to) zero at the surface ($r/R = 1$) compared to its central value.
- $\epsilon_{\rm nuc}$ becomes extremely small just outside the core ($r/R \lesssim 0.3$), and not at the surface.
- Therefore, C must be $T(r)$ (the surface value is $\sim 10^3$ K, compared to the maximum value at the centre $\sim 10^6$ K).

Therefore,

$$\boxed{\mathrm{A} \longrightarrow X(r)}$$ 2.0

$$\boxed{\mathrm{B} \longrightarrow l(r)}$$ 2.0

$$\boxed{\mathrm{C} \longrightarrow T(r)}$$ 2.0

No negative marking. Credit only for unique identification. No credit even for correct answer if the same letter is repeated for another quantity.

(T11.1b) What is the mass fraction of helium at the centre, Y_c, of the star? **3**

Solution:
Maximum value of X occurs outside the core, and is given by

$$X_s = 1 - Y_s - Z_s = 1 - 0.28 - 0.02 = 0.70$$ 1.0

Note Z remains constant throughout the star in the absence of gravitational settling and diffusion (given).
From the given graph, the central hydrogen mass fraction, X_c, is approximately

$$X_c \approx 0.55 X_s = 0.55 \times 0.70 = 0.39$$ 1.5

Credit of 1.0 for realising $X_c \neq 0.55$, but is normalised value.
Credit of 0.5 for correct readoff from graph $(0.53–0.55)$.
Then, the helium mass fraction at the centre is

$$Y_c = 1 - X_c - Z_c = 1 - 0.39 - 0.02 = \boxed{0.59}$$ 0.5

Half credit lower limit	Full credit range	Half credit upper limit
	0.59 to 0.61	

(T11.1c) Sketch the remaining two quantities from the list of five (which were not identified as curves A, B, or C) given in (T11.1a), as functions of r/R on the same graph in the Summary Answersheet, and label by their respective quantities. **5**

Solution:

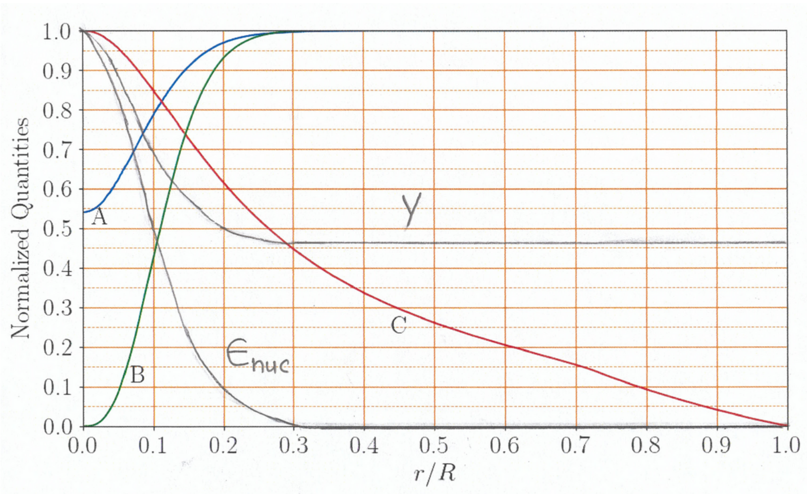
- The curve for $\epsilon_{\rm nuc}$ must mirror that of l. 2.5
  Any reasonable smooth curve between (0,1) and (0.3,0) is acceptable. Credit points:

Curve starts at (0,1)	0.5
Curve goes to zero between $(0.25, 0)$ and $(0.35, 0)$	1.0
Curve goes smoothly to zero	0.5

Curve is overall smooth	0.5

- The curve for Y must mirror that of X, and flatten outside the core to the normalized value corresponding to $Y_s = 0.28$.
  Previous result of $Y_c = 0.59$ gives this normalized value:
  $0.28/0.59 \approx 0.47$. 2.5
  Credit points:

Curve starts at (0,1)	0.5
Shape of curve mirrors that of X	0.5
The curve flattens between $r/R = 0.25$ and $r/R = 0.35$	0.5
The flattened value is between 0.45 and 0.50	1.0



For wrong identification of A, B, C in (T11.1a), student may draw curves of X, l or T in this part. Credit should be given for those as:

For X:

Curve starts between (0,0.5) and (0,0.6)	1.0
Shape of curve mirrors that of Y	0.5
The curve goes to 1.0 between $r/R = 0.25$ and $r/R = 0.35$	1.0

For l:

Curve starts at (0,0)	1.0
Shape of curve mirrors that of ϵ_{nuc}	0.5
The curve goes to 1.0 between $r/R = 0.25$ and $r/R = 0.35$	1.0

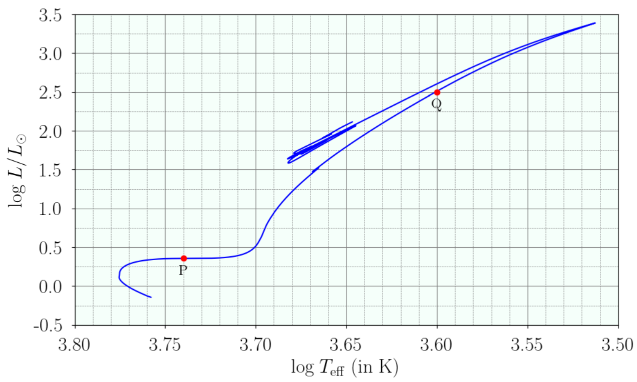
For T:

Curve starts at (0,1)	1.0
Curve decreases smoothly but not linearly	0.5
The curve goes to almost zero $r/R = 1.0$	1.0

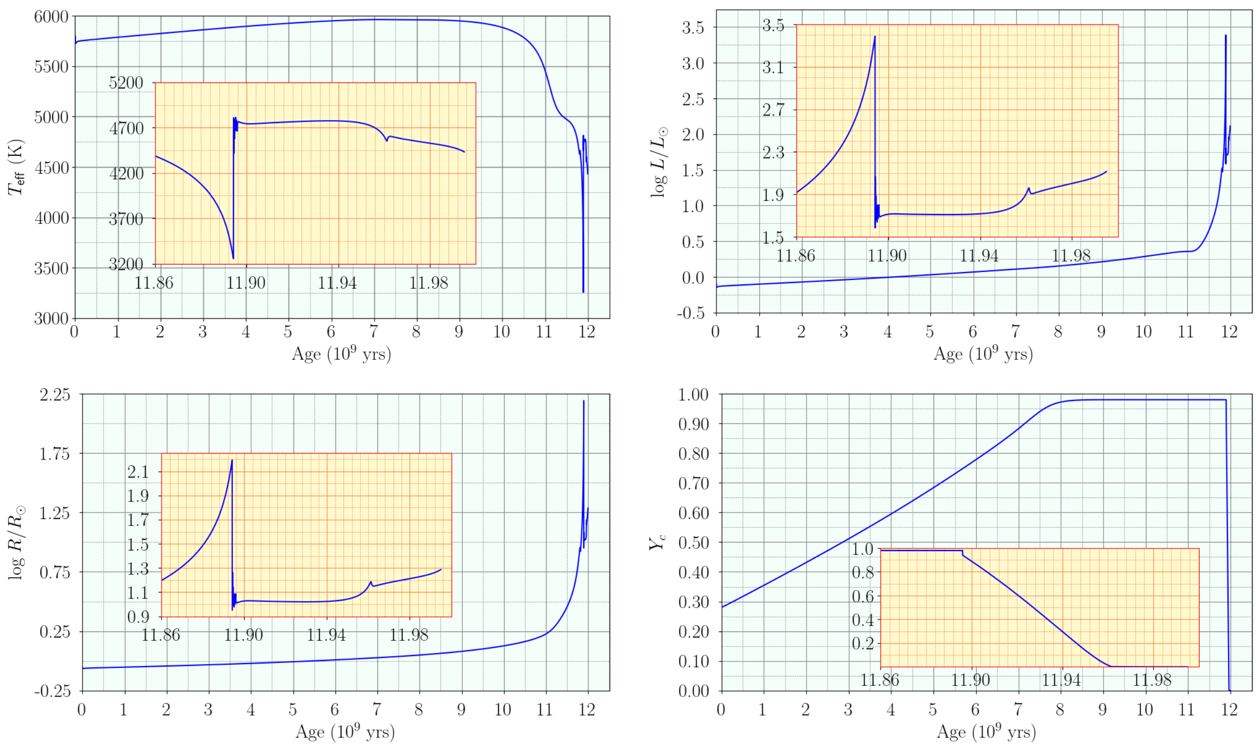
(T11.2) **Part 2: Evolving stars**

Consider the evolution of a $1\,\mathrm{M}_\odot$ star whose initial uniform composition is given by the mass fractions of helium, $Y_0 = 0.28$, and metals, $Z_0 = 0.02$. The figures below show the variation of different global quantities of this star as it evolves from ZAMS (Zero Age Main Sequence) till the end of helium burning in its core.

The graph below shows the evolutionary track of the star on the HR diagram (plot of $\log L/L_\odot$ vs $\log T_{\text{eff}}$, where L is the surface luminosity and T_{eff} is the effective temperature).



The figure below has four graphs which show the variation of T_{eff} (in K), L (plotted as $\log L/L_\odot$), R (plotted as $\log R/R_\odot$), and Y_c with age (in 10^9 yr) of the same star. In each of these four graphs, the insets show the variations of the respective quantities in detail between the ages of 11.86×10^9 yr to 12.00×10^9 yr, for greater clarity.



Use these graphs to answer the questions below.

(T11.2a) What is the approximate main sequence lifetime, t_{MS} (in years), of the star? **1**

Solution:
During the main sequence phase, the mass fraction of helium at the centre increases steadily, up to a maximum value of $Y_{\text{c, max}} = 1 - Z_{\text{c}}$. Here that

maximum value is reached at an age of around 8.5 Gyr, from the plot of Y_c vs age.

$$t_{MS} = 8.5 \times 10^9 \text{ yr}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	8.0×10^9 yr to 9.0×10^9 yr	

(T11.2b) What is the approximate duration, Δt_{He} (in years), for which the star burns helium in its core? **1**

Solution:
During the helium burning phase, the mass fraction of helium at the centre decreases steadily, down to 0. The inset in the plot of Y_c vs age shows that this corresponds to ages between 11.896×10^9 yr and 11.960×10^9 yr, corresponding to a duration of 64 Myr.

$$\Delta t_{He} = 64 \times 10^6 \text{ yr}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	62×10^6 yr to 66×10^6 yr	

(T11.2c) What fraction, f_H, of the initial amount of hydrogen at its centre has been burnt when the luminosity of the star is $1\,L_\odot$? **3**

Solution:
Read-off from the plot of $\log L/L_\odot$ vs age, we get age $\approx 4 \times 10^9$ yr. 1.0
At this age, read-off from plot of Y_c vs age, we get

$$Y_c \approx 0.60 \implies X_c \approx 1 - 0.60 - 0.02 = 0.38$$ 1.0

Therefore the fraction of original H burnt is

$$f_H = 1 - \frac{X_c}{X_0} = 1 - \frac{X_c}{1 - Y_0 - Z_0} = \boxed{0.46}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
0.39	0.45 to 0.46	0.53

Expression in % or fraction is ok.

(T11.2d) What is the radius of the star, R_1 (in units of $R_\odot$) when 60% of the initial amount of hydrogen at its centre has been burnt? **3**

Solution:

$$X_0 = 1 - Y_0 - Z_0 = 1 - 0.28 - 0.02 = 0.70$$

At this stage 60% of H has been burnt at the centre. So $X_c = 0.40 X_0 = 0.28$. Therefore,

$$Y_c = 1 - X_c - Z_c = 1 - 0.28 - 0.02 = 0.70$$ 2.0

From the graph of Y_c vs age, this corresponds to an age of approximately 5×10^9 yr. At this age, the value of radius is approximately $1\,R_\odot$ ($\log R/R_\odot \approx 0$), from the graph of $\log R/R_\odot$ vs age.

$$R_1 = 1\,R_\odot$$ 1.0

Any other answer is unlikely since it would involve eye approximation of reading of graphs, that is not to be credited.

(T11.2e) What are the radii of the star, R_P and R_Q (in units of $\mathrm{R}_\odot$), corresponding to its positions P and Q, respectively, as marked on the HR-diagram? **4**

Solution:
For P:
$\log T_\mathrm{eff} = 3.74 \implies T_\mathrm{eff} \approx 5500\,\mathrm{K}$ (the L, being flat, does not help here). 1.0
Read-off from plot of T_eff vs age, we get age $\approx 11 \times 10^9$ yr.
At this age, read-off from plot of $\log R/R_\odot$ vs age, we get $\log R/R_\odot \approx 0.25 \implies \boxed{R_\mathrm{P} = 1.78\,\mathrm{R}_\odot}$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$1.78\,\mathrm{R}_\odot$ to $1.80\,\mathrm{R}_\odot$	

Only credit of 0.5 in final step if left as $\log R/R_\odot = 0.25$.

For Q:
$\log L/L_\odot = 2.5$. (One can also use $\log T_\mathrm{eff} = 3.60 \implies T_\mathrm{eff} \approx 3981\,\mathrm{K}$ which would require rounding off to 3950 K at the next step.) 1.0
Read-off from (the inset) plot of $\log L/L_\odot$ vs age, we get age $\approx 11.884 \times 10^9$ yr. Notice that this value of $\log L/L_\odot$ occurs twice, and one has to take the first one, i.e., the ascending RGB branch.
At this age, read-off from (inset) plot of $\log R/R_\odot$ vs age, we get $\log R/R_\odot \approx 1.6 \implies \boxed{R_\mathrm{Q} = 39.81\,\mathrm{R}_\odot}$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$39.8\,\mathrm{R}_\odot$ to $40.0\,\mathrm{R}_\odot$	

Only credit of 0.5 in final step if left as $\log R/R_\odot = 1.6$.
Answer should be same if using T_eff in the first step.

(T11.3) **Part 3: Mass distribution inside a star**

The equation that governs the distribution of mass inside a star is given by

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

It would be convenient to express this equation in terms of three dimensionless variables, namely, the fractional mass, q, the fractional radius, x, and the relative density, σ, that we define as

$$q = m/M \qquad x = r/R \qquad \sigma = \rho/\bar{\rho}$$

where M and R are the total mass and radius of the star, respectively, and $\bar{\rho} \equiv \dfrac{M}{\frac{4}{3}\pi R^3}$ is the average density of the star. For the particular star that we shall be considering in this part, the following information is given:

- The central density $\rho(x=0) = 80\bar{\rho}$
- Half of the star's mass is contained within the inner 25% of its total radius, and 70% of its mass is contained within the inner 35% of its total radius.

In all subsequent parts of this question, it will be sufficient to round off all derived numerical coefficients to within 0.005.

(T11.3a) Express the above equation describing the dependence of mass on radius in terms of x, $\frac{dq(x)}{dx}$ and $\sigma(x)$. **2**

Solution:

$$\frac{dq}{dx} = \frac{dq}{dm}\frac{dm}{dr}\frac{dr}{dx}$$ 0.5

$$= \frac{1}{M}\left[4\pi(Rx)^2\rho\right] R$$

$$= 3\left(\frac{4\pi R^3}{3M}\right)x^2\rho = 3\frac{1}{\bar{\rho}}x^2\rho$$ 1.0

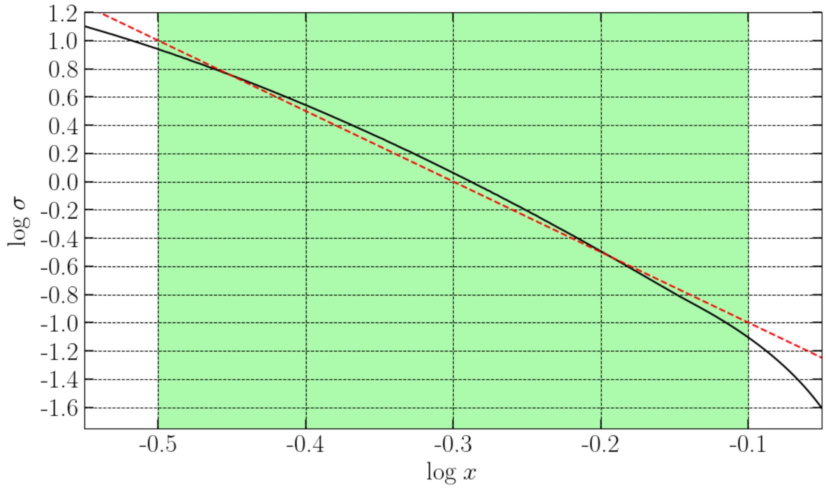
$$\therefore \boxed{\frac{dq}{dx} = 3x^2\sigma}$$ 0.5

To obtain the distribution of mass with radius, we need to know the density profile inside the star. For the purpose of this problem, we shall describe the variation of density with radius by approximate forms in two domains of x:

- the inner part of the star: $0 \le x \le 0.32$
- the middle part of the star: $0.32 < x < 0.80$

We do not make any approximation for the outermost part, i.e., $0.80 \le x \le 1.00$.

(T11.3b) **Approximation for the middle part**: The variation of $\log\sigma$, as a function of $\log x$ in the middle part of the star is shown (by the black curve) in the graph below. We shall make a linear approximation (shown as a dashed red line in the graph) for $\log\sigma$ as a function of $\log x$ in the domain $-0.5 < \log x < -0.1$, i.e., $0.32 \lesssim x \lesssim 0.80$ (shown by the green shaded domain). Further, we shall approximate the slope of this line by the nearest integer. **4**



Use this approximation to write an expression for $\sigma(x)$ as a function of x in the domain $0.32 < x < 0.80$.

Solution:
Let us denote the line as

$$\log\sigma = \alpha\log x + \beta$$

From the graph, we use the two end points $(-0.5, 1.0)$ and $(-0.1, -1.0)$. Therefore,

$$1.0 = -0.5\alpha + \beta$$

$-1.0 = -0.1\alpha + \beta$

Solving, we get $\alpha = -5$ and $\beta = -1.5$. Therefore,

$$\log\sigma = -5\log x - 1.5 \qquad \text{for } -0.5 < \log x < -0.1$$ 2.0

$$\therefore \sigma(x) = 10^{-1.5}x^{-5} \qquad \text{for } 0.32 < x < 0.80$$

$$\implies \boxed{\sigma(x) = 0.030x^{-5}} \qquad \text{for } 0.32 < x < 0.80$$ 2.0

Not rounding off $10^{-1.5}$ is acceptable with full credit.

(T11.3c) Use the result of (T11.3b) to derive an expression for $q(x)$ in the domain $0.32 < x < 0.80$. **6**

Solution:

$$\frac{dq}{dx} = 3x^2\sigma = 3x^2(0.030x^{-5}) = 0.090x^{-3}$$ 1.0

Integrating, we get

$$q(x) = \int 0.090x^{-3}dx$$ 1.0

$$= -\frac{0.045}{x^2} + C \qquad (C \text{ is a constant of integration})$$ 1.0

Without the constant of integration, only 0.5 marks is given in the last step. Now, given that $q(x = 0.35) = 0.7$. Using this, we get $C = 1.067$. 2.0

Therefore (rounding off to within 0.005),

$$\boxed{q(x) = 1.065 - \frac{0.045}{x^2} \qquad \text{for } 0.32 < x < 0.80}$$ 1.0

If consistent with rounding off at earlier steps, values between 1.065 and 1.091 for the constant term are acceptable with full credit.

(T11.3d) **Approximation for the inner part**: In the inner part of the star ($0 \leq x \leq 0.32$), the density may be approximated as a linear function of radius, i.e., $\sigma(x) = Ax + B$, where A, B are constants. Determine A and B, and hence obtain an expression for $q(x)$ in the domain $0 \leq x \leq 0.32$. Note that the approximations adopted in the previous part and this part may lead to small discontinuities in density or mass at $x = 0.32$. **8**

Solution:

$$\frac{dq}{dx} = 3x^2\sigma = 3x^2(Ax + B) = 3Ax^3 + 3Bx^2$$ 1.0

Integrating, we get

$$q(x) = \int (3Ax^3 + 3Bx^2)dx$$ 1.0

$$= \frac{3A}{4}x^4 + Bx^3 + D \qquad (D \text{ is a constant of integration})$$ 1.0

Without the constant of integration, only 0.5 marks is given in the last step.

Three boundary conditions applied sequentially to find the constants:

- Mass at the centre is zero, i.e., $q(0) = 0 \implies D = 0$ 0.5

- $\rho(0) = 80\bar{\rho}$ (given), i.e., $\sigma(0) = 80 \implies \boxed{B = 80}$ 1.5
- Given $q(0.25) = 0.50$. Therefore,

$$0.5 = \frac{3A}{4}(0.25)^4 + 80$$
$$\implies A = \frac{4}{3}\frac{0.50 - 80(0.25)^3}{(0.25)^4}$$
$$\implies \boxed{A = -256}$$ 2.0

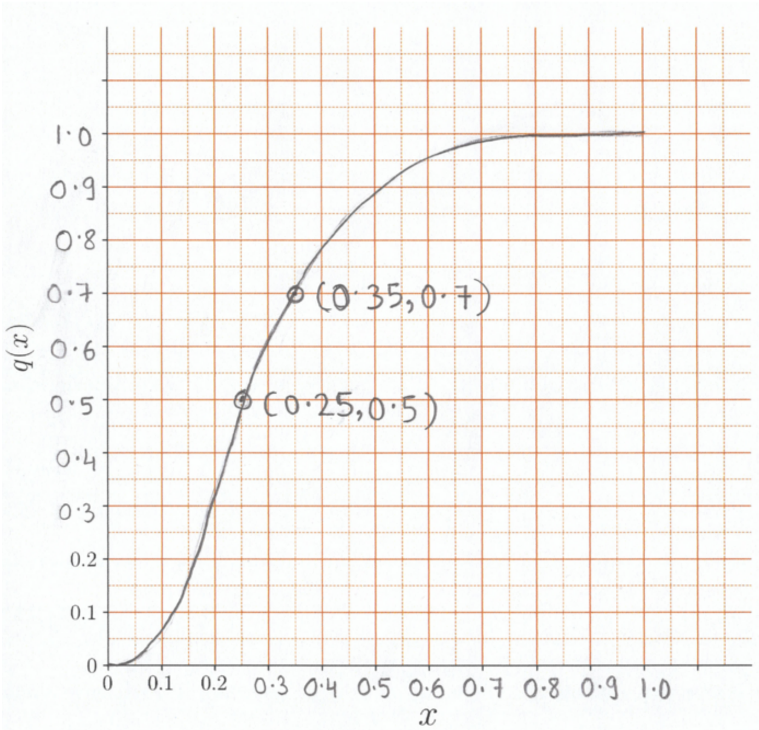
Therefore, putting the values of A and B,

$\boxed{q(x) = -192x^4 + 80x^3 \quad \text{for } 0 \le x \le 0.32}$ 1.0

Using continuity at $x = 0.32$ instead of either of the above conditions will not be credited, since discontinuity has been indicated in the question.

(T11.3e) The expressions for $q(x)$ obtained in parts (T11.3c) and (T11.3d) are approximations that describe the variation of mass with radius quite well, but only in specific regions of the star. For the domain $0.80 \le x \le 1$ (for which we have not derived any expression), it is possible to use appropriate extrapolation from the neighbouring region. Use these approximate expressions and given data to sketch a smooth curve (without any discontinuities either in $q(x)$ or its derivative) for $q(x)$ vs x for the entire star ($0 \le x \le 1$) that represents the variation of mass with radius. 4

Solution:



Credit will be given on the following aspects:

Curve begins at (0,0) and ends at (1,1)	0.5
Slope is zero at both ends	1.0

Curve passes through the two given points	1.0
Initial part of the curve looks like a cubic-quartic	0.5
Latter part of the curve looks negative inverse-square with a dc shift	0.5
Curve is overall smooth	0.5

(T12) Hawking Radiation from Black Holes **[50 marks]**

(T12.1) Stellar mass blackholes

A black hole (BH) typically forms by the gravitational collapse of a massive star at the end of its life cycle to a point called a singularity. Due to the extreme gravity of such an object, nothing that enters the so-called event horizon (a spherical surface with $r = R_{SC}$, where r is the distance from the singularity) is able to escape from it. Here, R_{SC} is referred to as the Schwarzschild radius.

(T12.1a) **Modelling the origin of Hawking radiation:** Consider a pair of particles, each with mass m, produced on either side of the BH horizon. One particle is slightly outside the horizon at $r \approx R_{SC}$, while the other particle is inside the horizon at $r = \kappa R_{SC}$. Assume that the total energy of a particle is the sum of its rest mass energy mc^2 and the gravitational potential energy due to the BH. Determine the value of κ for which the particle pair has zero total energy. **4**

Solution:

- Total energy of any particle is $E = E_{grav} + mc^2$, so we have
$$E_1 = -\frac{GM_{bh}m}{R_{SC}} + mc^2 , \qquad E_2 = -\frac{GM_{bh}m}{\kappa R_{SC}} + mc^2 .$$ 1.0
- We require $E_1 + E_2 = 0$. Hence,
$$2mc^2 - \frac{GM_{bh}m}{R_{SC}}\left(1 + \frac{1}{\kappa}\right) = 0$$ 1.0
- Then
$$\frac{1}{\kappa} = \frac{2c^2 R_{SC}}{GM_{bh}} - 1 = 3$$
where we have used $R_{SC} = \dfrac{2GM_{bh}}{c^2}$. 1.0
- This gives $\boxed{\kappa = \frac{1}{3}}$. 1.0

(T12.1b) **Temperature of a black hole:** If the particle produced outside the horizon in the above process has enough kinetic energy, it may escape the BH in a process called Hawking radiation. The one inside the horizon, which has negative energy, gets absorbed and decreases the mass of the BH. **4**

Assume that all Hawking radiation is made of photons with a black body spectrum which peaks at the wavelength $\lambda_{bb} \approx 16R_{SC}$. It is known that for a solar mass BH, $R_{SC,\odot} = 2.952\,\mathrm{km}$.

Obtain an expression for the temperature, T_{bh}, of the BH corresponding to this black body radiation, in terms of its mass M_{bh} and physical constants. Calculate the Schwarzschild radius, $R_{SC,10\odot}$, and temperature, $T_{bh,10\odot}$, for a BH with mass $10\,\mathrm{M}_\odot$.

Solution:

- Using Wien's law, $\lambda_{bb} = b/T_{bh}$ where $b = 2.898 \times 10^{-3}\,\mathrm{m\,K}$‘. 1.0
- Hence, $$T_{bh} = \frac{b}{16R_{SC}} \implies \boxed{T_{bh} = \frac{bc^2}{32GM_{bh}}}$$ 1.0

- Since R_{SC} for a solar mass black hole is 2.952 km,

  For a black hole with mass $= 10\ M_\odot$, we get $\boxed{R_{\text{SC},\,10\odot} \simeq 29.52\,\text{km}}$. 1.0

  Given the significant figures, 29 or 30 km will get full marks.

- The temperature of this BH will be

$$T_{\text{bh},\,10\odot} = \frac{2.898\times10^{-3}\text{m K}}{16\times R_{\text{SC},\,10\odot}}$$

$$\boxed{T_{\text{bh},\,10\odot} = 6.1\times10^{-9}\,\text{K}}\ .$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$6.0\times10^{-9}\,\text{K}$ to $6.2\times10^{-9}\,\text{K}$	

(T12.1c) **Mass loss of a black hole:** Assume that the Hawking radiation is emitted out from the event horizon. Using the mass-energy equivalence, obtain an expression for the rate of mass loss, $dM_{\text{bh}}(t)/dt$, in terms of the mass $M_{\text{bh}}(t)$ of the BH and physical constants. 8

Hence, obtain an expression for $M_{\text{bh}}(t)$ for a BH with initial mass M_0. Sketch $M_{\text{bh}}(t)$ as a function of t from $M_{\text{bh}} = M_0$ to $M_{\text{bh}} = 0$.

Solution:

- Power emitted by an evaporating black hole $= \sigma T_{\text{bh}}^4 4\pi R_{\text{SC}}^2$. 1.0

- This gives

$$\frac{dM_{\text{bh}}}{dt} = -\frac{1}{c^2}\sigma T_{\text{bh}}^4 4\pi R_{\text{SC}}^2\,.$$ 1.0

- With $T_{\text{bh}} = b/(16R_{\text{SC}})$,

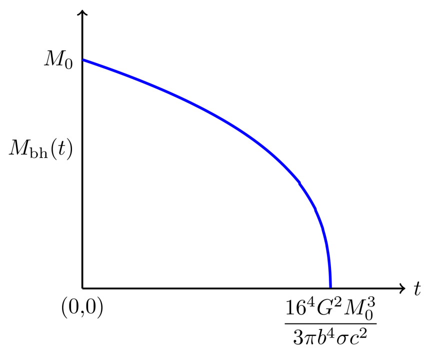
$$\boxed{\frac{dM_{\text{bh}}}{dt} = -\frac{\pi b^4\sigma c^2}{16^4G^2}\frac{1}{M_{\text{bh}}^2}}$$ 1.0

- This gives $\dfrac{dM_{\text{bh}}}{M_{\text{bh}}^2} = -\dfrac{\pi b^4\sigma c^2}{16^4G^2}dt$, and hence $\dfrac{M_{\text{bh}}^3}{3} = -\dfrac{\pi b^4\sigma c^2}{16^4G^2}t + C$. 1.0

- Since $M_{\text{bh}} = M_0$ at $t = 0$, we get $M_{\text{bh}}^3 = M_0^3 - 3\dfrac{\pi b^4\sigma c^2}{16^4G^2}t$ 0.5

$$\boxed{M_{\text{bh}}(t) = \left(M_0^3 - 3\frac{\pi b^4\sigma c^2}{16^4G^2}t\right)^{1/3}}$$ 1.0

The value of $\dfrac{\pi b^4\sigma c^2}{16^4G^2} = 3.868\times10^{15}\,\text{kg}^3\,\text{s}^{-1}$, not necessary here, but gets full credit.



2.5

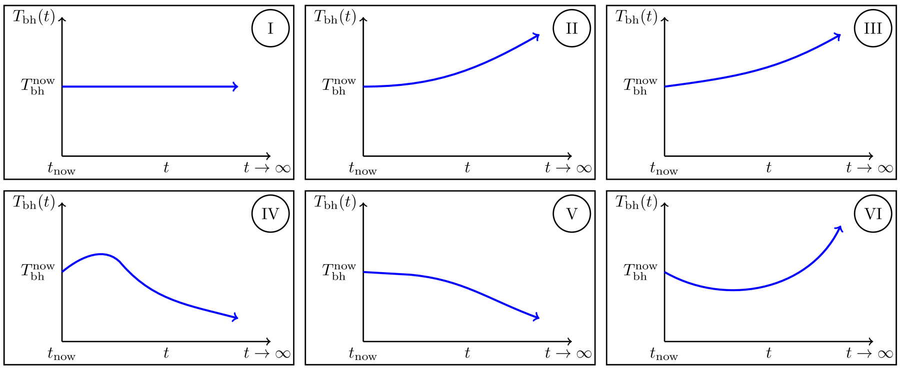
M_0 marked on $M_{\rm bh}(t)$-axis	0.5
Correct expression for the intercept on the t-axis	0.5
Negative initial slope	0.5
Large negative final slope	0.5
Convex nature of the curve	0.5

(T12.1d) **Lifetime of a black hole:** Obtain an expression for the lifetime $\tau_{\rm bh}$ at which a black hole with initial mass M_0 completely evaporates due to Hawking radiation, in terms of M_0 and physical constants. Calculate the lifetime $\tau_{{\rm bh},10\odot}$ (in seconds) for a black hole with $M_0 = 10\,{\rm M}_\odot$. **3**

Solution:

- We know that $M(t) = \left(M_0^3 - 3\frac{\pi b^4 \sigma c^2}{16^4 G^2} t\right)^{1/3}$.
- Clearly, $M(t) = 0$ when $t = \frac{M_0^3}{3}\frac{16^4 G^2}{\pi b^4 \sigma c^2}$. Thus, $\boxed{\tau_{\rm bh} = \frac{M_0^3}{3}\frac{16^4 G^2}{\pi b^4 \sigma c^2}}$ 1.0
- With $A \equiv \frac{\pi b^4 \sigma c^2}{16^4 G^2} \simeq 3.868 \times 10^{15}\,{\rm kg^3\,s^{-1}}$, this gives 0.5
$$\tau_{\rm bh} \approx \frac{M_0^3}{3A} = 8.617 \times 10^{-17}\left(\frac{M_0}{\rm kg}\right)^3 \,{\rm s}.$$ 0.5
- For $M_0 = 10\,{\rm M}_\odot$,
$$\boxed{\tau_{{\rm bh},10\odot} \approx 6.8 \times 10^{77}\,{\rm s}}$$ 1.0

(T12.1e) **Black hole in a CMB radiation bath:** Consider an isolated black hole in space, far away from other bodies, with a current temperature $T_{\rm bh}^{\rm now}$, surrounded by the cosmic microwave background (CMB) with a current temperature $T_{\rm cmb}^{\rm now} = 2.7\,{\rm K}$. The black hole can grow in mass by absorbing CMB radiation, and lose its mass by Hawking radiation. **6**
Taking into account the accelerating expansion of the Universe, identify which of the following figures show the long-term time evolution of $T_{\rm bh}$ in the following three cases:
(X) $T_{\rm bh}^{\rm now} > T_{\rm cmb}^{\rm now}$, (Y) $T_{\rm bh}^{\rm now} = T_{\rm cmb}^{\rm now}$, (Z) $T_{\rm bh}^{\rm now} < T_{\rm cmb}^{\rm now}$.



Indicate your answer by ticking the appropriate box (only one) for each case X, Y or Z in the Table given in the Summary Answersheet corresponding to the appropriate figure number.

Solution:
As the CMB will redshift away eventually, there will be no CMB radiation for the BH to absorb at large t. It will only decrease in mass and increase in temperature. So only the figures which show the temperature of the black hole increasing eventually ($t \to \infty$) are relevant. The derivatives at the origin (now) should distinguish the three cases.

(X) If $T_{\rm bh}^{\rm now} > T_{\rm cmb}^{\rm now}$, then the BH is losing more energy now than it is gaining by CMB absorption, and its temperature should increase due to decreasing mass. $(dT/dt)_{\rm now} > 0$.

(Y) If $T_{\rm bh}^{\rm now} = T_{\rm cmb}^{\rm now}$, the temperature will be in equilibrium for some time, i.e. $(dT/dt)_{\rm now} = 0$, before the CMB temperature redshifts to smaller values and $(dT/dt)_{\infty} > 0$.

(Z) For $T_{\rm bh}^{\rm now} < T_{\rm cmb}^{\rm now}$, initially the BH will gain mass (and hence cool down) by absorbing from CMBR compared to what it loses by evaporation. So $(dT/dt)_{\rm now} < 0$. But eventually, $(dT/dt)_{\infty} > 0$.

	I	II	III	IV	V	VI
X			$\surd$			
Y		$\surd$				
Z						$\surd$

2.0 marks each for one and only one correct tick for the cases X, Y, Z. More than one tick for any case receives no credit for that case. 6.0

(T12.2) Primordial black holes (PBHs) of much smaller masses can form in the very early Universe. All the following questions are related to PBHs. Here, any processes that increase the mass of the black hole may be neglected.

(T12.2a) **PBH evaporating at the current epoch:** As you may have noticed from the answers to the previous questions, black holes of solar mass would take a long time to evaporate. However, since PBHs can have a much smaller mass, we may be able to see them evaporating in current times. 4
Find the initial mass $M_{0,\,\rm PBH}$ (in kg), Schwarzschild radius $R_{\rm SC,\,PBH}$ (in m), and temperature $T_{\rm PBH}$ (in K) of a black hole that may be evaporating away completely at the present epoch, i.e., those with lifetime $\tau_{\rm PBH} = 14$ billion years.

Solution:

- We want $\tau_{\text{PBH}} = 14 \times 10^9 \times 3.156 \times 10^7\,\text{s} = 4.418 \times 10^{17}\,\text{s}$. 1.0
- But we have found in (T12.1d) that $\tau_{\text{bh}} = 8.617 \times 10^{-17} \left(\frac{M_0}{\text{kg}}\right)^3\,\text{s}$.

  This gives $M^3_{0,\,\text{PBH}} = \left(\frac{\tau_{\text{PBH}}}{8.617 \times 10^{-17}}\right)\,\text{kg}^3$

  $\Rightarrow$ $M_{0,\,\text{PBH}} \simeq 1.7 \times 10^{11}\,\text{kg}$ 1.0
- Since $R_{\text{SC}} \propto M_{\text{bh}}$, and we know $R_{\text{SC},\,\odot} = 2952\,\text{m}$,

  $R_{\text{SC, PBH}} \simeq R_{\text{SC},\,\odot}\left(\frac{M_{0,\,\text{PBH}}}{M_\odot}\right) \Rightarrow$ $R_{\text{SC, PBH}} = 2.6 \times 10^{-16}\,\text{m}$ 1.0
- Since $T_{\text{bh}} \propto 1/M_{\text{bh}}$ and we know $T_{\text{bh},\,10\odot}$ from (T12.1b),

  $T_{\text{PBH}} \simeq T_{\text{bh},\,10\odot}\left(\frac{10M_\odot}{M_{0,\,\text{PBH}}}\right) \Rightarrow$ $T_{\text{PBH}} = 7.1 \times 10^{11}\,\text{K}$ 1.0

(T12.2b) **Formation of a PBH:** In the radiation-dominated early Universe, the scale factor varies as $a(t) \sim t^{1/2}$. In this era, PBHs form due to the collapse of all energy contained in a region of physical size ct, where t is the age of the Universe at that time. **6**
A PBH with mass of 10^{12} kg forms when the age of the Universe is about 10^{-23} s. Calculate the age of the Universe, t_{20}, when a PBH of mass 10^{20} kg forms.

Solution:

- The size of horizon $r_{\text{H}} = ct$. So the total mass contained within the horizon volume is
$$M = \frac{1}{c^2}\rho\left(\frac{4\pi}{3}\right)\left(\frac{ct}{2}\right)^3.$$ 2.0
- With $a(t) \sim t^{1/2}$ and $a(t) \sim 1/T$, one gets $t \sim T^{-2}$. 2.0
- With $\rho \sim T^4 \sim t^{-2}$, the mass of PBH formed at time t would be
$$M_{0,\,\text{PBH}} \propto t$$
  . 1.0
- The age of the Universe at which PBH of 10^{12} kg forms is $t_{12} = 10^{-23}$ s.
- Hence, $t_{20} = t_{12} \times \left(\frac{10^{20}\,\text{kg}}{10^{12}\,\text{kg}}\right) \Rightarrow$ $t_{20} = t_{12} \times 10^8 = 10^{-15}\,\text{s}$ 1.0

(T12.2c) **Observed spectrum of Hawking radiation from PBH:** Consider a PBH of initial mass 10^{10} kg which completely evaporates at the end of its lifetime τ_{PBH}. For this part, assume for simplicity that most of the Hawking radiation is emitted at this time, with a temperature corresponding to its initial mass. Also, take the scale factor of the Universe to be evolving as $a(t) \sim t^{2/3}$. Calculate the peak wavelength of this Hawking radiation as observed at Earth, λ_{earth}, at the present epoch (at $t = 14$ billion years). **5**

Solution:

- Lifetime of the PBH with $M_{0,\,\mathrm{PBH}} = 10^{10}$ kg will be
$$\tau_{\mathrm{PBH}} = 8.617 \times 10^{-17} \cdot (10^{10})^3 \mathrm{s} = 8.617 \times 10^{13}\,\mathrm{s}\,.$$ 0.5

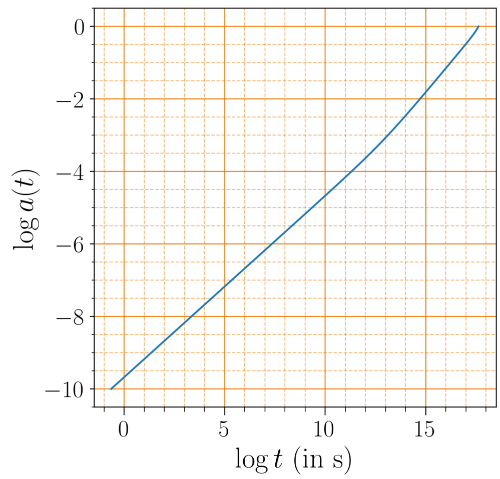
- As given in (T12.1b), $R_{\mathrm{SC},\odot} = 2952\,\mathrm{m}$. Using $R_{\mathrm{SC}} \propto M$, we get the peak wavelength of radiation emitted by this 10^{10} kg PBH, at the time of emission, as
$$\lambda_0 \simeq 16 R_{\mathrm{SC}} = 16 \cdot \frac{10^{10}\mathrm{kg}}{M_\odot} \cdot R_{\mathrm{SC},\odot} = 2.4 \times 10^{-16}\,\mathrm{m}.$$ 1.5

- Redshifted wavelength received at earth at present time t_0 is obtained from
$$\lambda_{\mathrm{earth}} = \lambda_0 \frac{a(t_0)}{a(\tau_{\mathrm{PBH}})}.$$ 1.0

- With $a(t) \sim t^{2/3}$, we get
$$\lambda_{\mathrm{earth}} = \lambda_0 \left(\frac{t_{\mathrm{now}}}{\tau_{\mathrm{PBH}}}\right)^{2/3} = 2.4 \times 10^{-16}\,\mathrm{m} \left(\frac{4.418 \times 10^{17}\,\mathrm{s}}{8.617 \times 10^{13}\,\mathrm{s}}\right)^{2/3}$$ 1.0
$$\boxed{\lambda_{\mathrm{earth}} = 7.1 \times 10^{-14}\,\mathrm{m}}$$ 1.0

(T12.2d) **High energy cosmic radiation from PBH:** Now assume that the Hawking radiation emitted at a given time t corresponds to photons emitted with an energy $k_{\mathrm{B}} T_{\mathrm{bh}}(t)$. Also, the highest possible temperature for a black hole is the Planck temperature T_{Planck} where $k_{\mathrm{B}} T_{\mathrm{Planck}} = 10^{19}$ GeV. **10**

The evolution of the scale factor over relevant time scales is given in the following figure. The scale factor today is set to be unity. $t(s)$ on the time axis represents the age of the universe in seconds.



If a photon with an energy of $E_{\mathrm{det}} = 3.0 \times 10^{20}$ eV is observed on Earth, determine the largest and the smallest possible values of the initial mass of the PBH ($M_0^{\max}$ and $M_0^{\min}$, respectively) which could be responsible for this photon.

Solution:

- Using the expression for T_{bh} found in (T12.1b), the energy of a typical photon emitted at time t for a PBH is

$$E_{\text{emitted}}(t) = k_B T_{\text{bh}}(t) = \frac{bc^2 k_B}{32 G M_{\text{bh}}(t)}.$$ 1.0

- From (T12.1c), we know

$$M_0^3 = 3\frac{\pi b^4 \sigma c^2}{16^4 G^2} t + M_{\text{bh}}(t)^3 = 3At + \left(\frac{bc^2 k_B}{32 G E_{\text{emitted}}(t)}\right)^3.$$ 1.0

- The value of E_{det} can be different from E_{emitted} due to redshift. Hence, the smallest possible value of $E_{\text{emitted}}(t)$ is $E_{\text{det}} = 3.0 \times 10^{20}$ eV. This gives the maximum value of the second term as

$$\left(\frac{bc^2 k_B}{32 G E_{\text{emitted}}(t)}\right)^3_{\text{max}} = 4.3 \times 10^{-5}\,\text{kg}^3.$$ 1.0

- Given the value of $A = 3.868 \times 10^{15}\,\text{kg}^3\,\text{s}^{-1}$, the second term can always be neglected for $t \gtrsim 10^{-19}$ s. This is true in this problem. So $M_0 \approx (3At)^{1/3}$. 1.0

- **The largest possible** M_0 would be obtained when t is as large as possible, i.e., $t = t_{\text{now}}$. This will correspond to a BH radiating in its last stages at present in the nearby region. 1.0

- Then

$$M_0^{\text{max}} = (3At_{\text{now}})^{1/3} = (5.13 \times 10^{33}\text{kg}^3)^{1/3}.$$ 0.5

- Thus,

$$\boxed{M_0^{\text{max}} = 1.7 \times 10^{11}\,\text{kg}.}$$ 0.5

- Note that this was already found in (T12.2a), so the calculation above is not needed.

- **The smallest** M_0 will correspond to a BH that has evaporated as early as possible. Such a BH would have evaporated far away from Earth, and its radiation would be redshifted.

- The smallest time of evaporation would correspond to the largest redshift factor, and hence the largest $E_{\text{emitted}}(t)$, which is $E_{\text{emitted}}(t) = k_B T_{\text{Planck}}$. 1.0

- The ratio of the scale factors at the time of the above PBH evaporation and now is

$$\frac{a(t)}{a(t_{\text{now}})} = \frac{E_{\text{det}}}{k_B T_{\text{Planck}}} = \frac{3.0 \times 10^{20}\,\text{eV}}{10^{19}\,\text{GeV}} = 3.0 \times 10^{-8}$$

With $a(t_{\text{now}}) = 1.0$, we have $a(t) = 3.0 \times 10^{-8}$. 1.0

- Using the plot of $a(t)$ vs t from the figure, $t \simeq 10^4$ s. Note that for this t, the second term in the original M_0^3 expression is still negligible. 1.0

- Therefore, $\boxed{M_0^{\text{min}} \approx (3At)^{1/3} \approx 4.9 \times 10^6\,\text{kg}}$. 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	2.5×10^6 kg to 10.0×10^6 kg	