(D01) **30 Years of Exoplanets** **[90 marks]**

This problem explores some aspects of the two main methods of exoplanet detection: radial velocity and transit. Throughout this problem we shall consider a particular system of a single planet (P) in a circular orbit with radius a around a solar-type star (S). We shall refer to this system as the "SP system".

Stellar parameters

(D01.1) The V-band apparent magnitude of the star S is (7.65 ± 0.03) mag, the parallax is (20.67 ± 0.05) milliarcsecond and the bolometric correction (BC) is -0.0650 mag. Thus the star has a higher bolometric luminosity than its luminosity in the V-band. **8**

Estimate the mass of the star, $M_{\rm s}$ (in units of $\mathrm{M}_\odot$), assuming a mass-luminosity (M–L) relation of the form $L \propto M^4$. Also estimate the uncertainty in $M_{\rm s}$. You may need $\frac{d}{dx}\ln x = \frac{1}{x}$.

Solution:

Given

Parallax: $p = (20.67 \pm 0.05)$ milliarcsecond

V-band apparent magnitude: $m_{\rm V} = (7.65 \pm 0.03)$ mag

Let:

V-band absolute magnitude: $M_{\rm V}$

Bolometric Magnitude: $M_{\rm bol}$

Bolometric Magnitude of the Sun: $M_{\rm bol,\odot} = 4.74$ mag (given)

$$M_{\rm V} = m_{\rm V} + 5(\log p + 1) \qquad = 4.227\,\text{mag}$$

$$M_{\rm bol} = M_{\rm V} + BC$$

$$= m_{\rm V} + 5(\log p + 1) + \text{BC} = 4.162\,\text{mag}$$

$$\text{Also} \quad M_{\rm bol} = M_{\rm bol,\odot} - 2.5\log\frac{L_{\rm bol}}{L_\odot}$$

$$\therefore \quad L_{\rm bol} = 10^{\frac{M_{\rm bol,\odot} - M_{\rm bol}}{2.5}} L_\odot \qquad = 6.521 \times 10^{26}\,\text{W}$$

$$\text{Then} \quad M_{\rm s} = \left(\frac{L_{\rm bol}}{L_\odot}\right)^{1/4} M_\odot \qquad = 1.14\,\mathrm{M}_\odot$$

Uncertainty estimate:

$$\Delta M_{\rm bol} = \sqrt{(\Delta m_{\rm V})^2 + \left(\frac{5}{\ln 10}\frac{\Delta p}{p}\right)^2}$$

$$= 0.03\,\text{mag}$$

$$\text{Now,} \quad M_{\rm s} = 10^{\frac{M_{\rm bol,\odot} - M_{\rm bol}}{10}}$$

$$\therefore \quad \Delta M_{\rm s} = M_{\rm s}\frac{\Delta M_{\rm bol}}{10}\ln 10$$

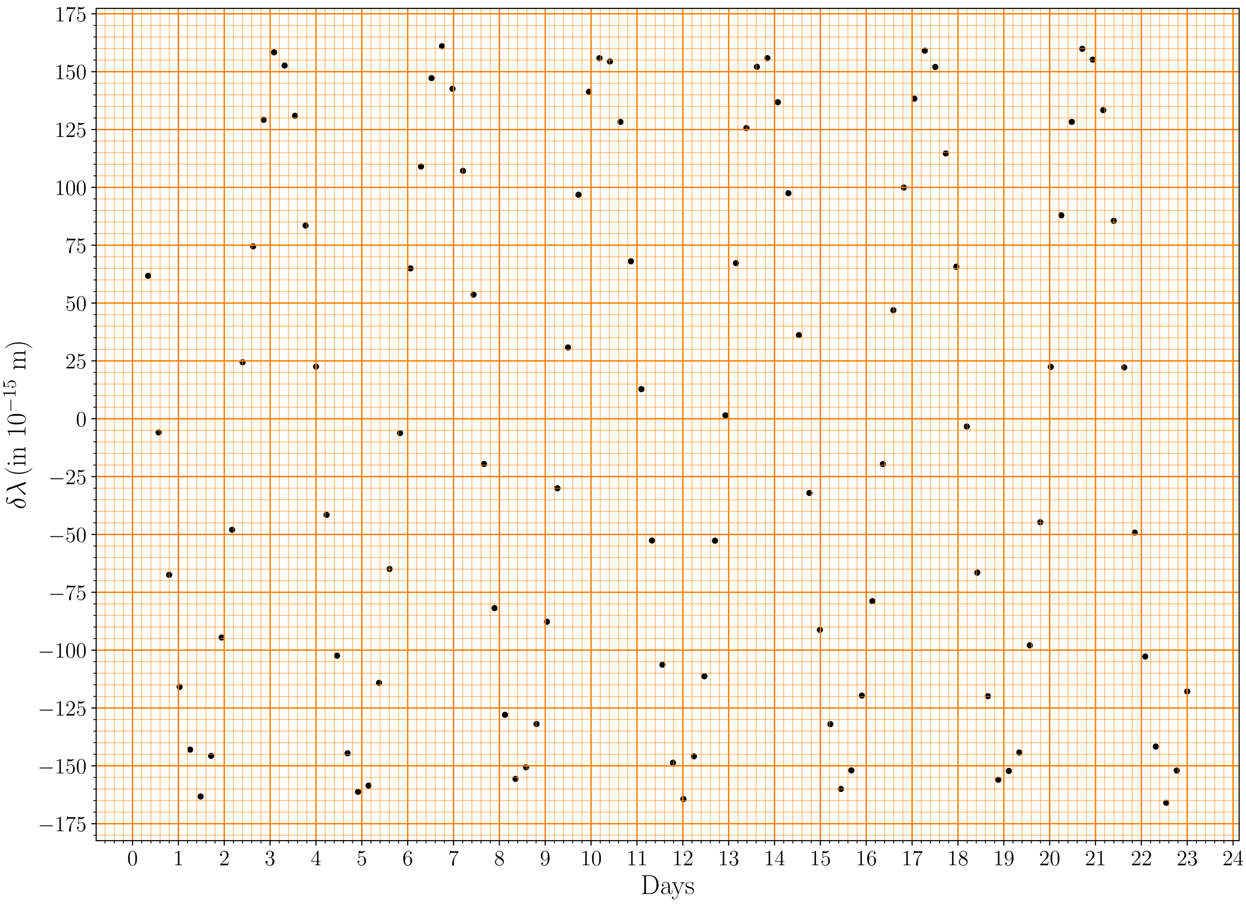
$$= 0.01\,\mathrm{M}_\odot$$

$$M_{\rm s} = (1.14 \pm 0.01)\,\mathrm{M}_\odot$$

Radial Velocity method

The radial velocity method uses the Doppler shift $\delta\lambda \equiv \lambda_{\text{obs}} - \lambda_0$ between the observed wavelength λ_{obs} and the rest wavelength λ_0 of a known spectral line to detect an exoplanet and determine its characteristics.

The figure below shows the $\delta\lambda$ for the Fe I line ($\lambda_0 = 543.45 \times 10^{-9}$ m) as a function of time as observed for the SP system.



The radial velocity semi-amplitude K is defined as $K \equiv (v_{\text{r, max}} - v_{\text{r, min}})/2$ where $v_{\text{r, max}}$ and $v_{\text{r, min}}$ are the minimum and maximum radial velocities, respectively. For a circular planetary orbit the semi-amplitude K can be written as:

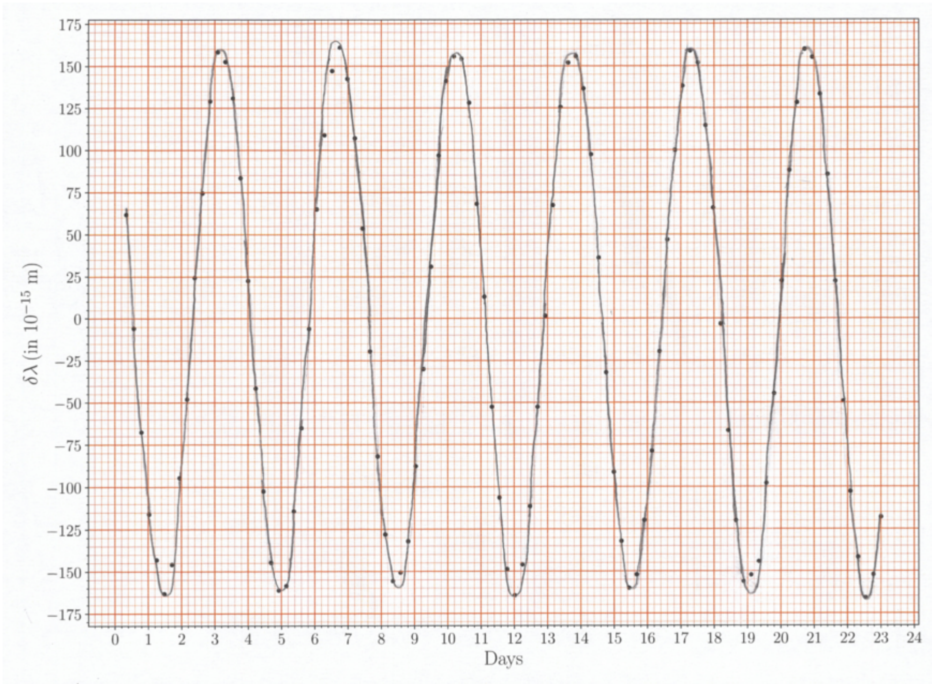
$$K = \left(\frac{2\pi G}{T}\right)^{1/3} \frac{M_{\text{p}} \sin i}{(M_{\text{p}} + M_{\text{s}})^{2/3}}$$

where T is the period, i is the inclination of the planetary orbit (angle between the normal to the orbital plane of the planet and the line of sight of the observer), M_{p} and M_{s} are the masses of the planet and the star, respectively.

(D01.2) Use the graph given in the Summary Answersheet to answer the following.

(D01.2a) Draw a smooth curve associated with the observed data shown in the graph. **2**

Solution:



(D01.2b) Select appropriate points on your drawn curve and use suitable methods to determine T and K along with respective uncertainties. All data points used for the calculation of T and K must be shown in the table in the Summary Answersheet. Use the rest of the table to show your intermediate calculations, as needed, with appropriate headers. **11**

Solution:

T Determination:

Least count on time axis = 0.2 d

There are 13 zero crossings (ZC) and 6 periods.

Difference between ZC-1 and ZC-13 = (21.6 − 0.6) = 21.0 d

$$T = \frac{21}{6} = 3.5\,\mathrm{d}$$

Uncertainty in T:

$$\Delta T = \frac{1}{6}\sqrt{(0.2^2 + 0.2^2)} = 0.05\,\mathrm{d}$$

$$T = (3.50 \pm 0.05)\,\mathrm{d}$$

K Determination:

Serial No. of Extrema	Values of Extrema ($\times 10^{-15}$m)	$2\delta\lambda$ m	K m s^{-1}
1	−165		
2	160	325	
3	−160		
4	165	325	
5	−160		
6	155	315	
7	−165		
8	155	320	
9	−160		
10	160	320	
11	−165		

Least count on $\delta\lambda$ axis $= 5 \times 10^{-15}\,\mathrm{m}$
Differences in measured extrema ($\delta\lambda$) of alternate pairs are written in table.

$$2(\delta\lambda)_{\mathrm{Mean}} = 321.67 \times 10^{-15}\,\mathrm{m}$$
$$(\delta\lambda)_{\mathrm{Mean}} = 160.83 \times 10^{-15}\,\mathrm{m}$$
$$K = \frac{\delta\lambda}{\lambda} \times c = \frac{160.83 \times 10^{-15} \times 2.998 \times 10^{8}}{543.45 \times 10^{-9}} = 88.73\,\mathrm{m\,s^{-1}}$$

Uncertainty in K:

$$\sigma_{\delta\lambda} = \sqrt{\frac{1}{N-1}\Sigma\left(\delta\lambda - (\delta\lambda)_{\mathrm{Mean}}\right)^2} = 2.041 \times 10^{-15}\,\mathrm{m}$$
$$\sigma_K = \frac{\sigma_{\delta\lambda}}{\lambda} \times c = \frac{2.041 \times 10^{-15} \times 2.998 \times 10^{8}}{543.45 \times 10^{-9}} = 1.12\,\mathrm{m\,s^{-1}}$$
$$K = (89 \pm 1)\,\mathrm{m\,s^{-1}}$$

(D01.2c) Find the minimum mass of the planet $M_{\mathrm{p,\,min}}$ (in $\mathrm{M_\odot}$), and its corresponding uncertainty, assuming $M_{\mathrm{p}} \ll M_{\mathrm{s}}$. **5**

Solution:
The minimum mass of the planet $M_{\mathrm{p,\,min}} \equiv M_{\mathrm{p}} \sin i$ comes from equation of K.

$$K = \left(\frac{2\pi G}{T}\right)^{1/3} \frac{M_{\mathrm{p}} \sin i}{(M_{\mathrm{p}} + M_{\mathrm{s}})^{2/3}}$$

We assume $M_{\mathrm{p}} + M_{\mathrm{s}} \approx M_{\mathrm{s}}$.

For $M_{\mathrm{s}} = 1.14\,\mathrm{M_\odot}, T = 3.5\,\mathrm{d}$:

$$M_{\mathrm{p,\,min}} \approx K M_{\mathrm{s}}^{2/3} \left(\frac{T}{2\pi G}\right)^{1/3} = 6.927 \times 10^{-4}\,\mathrm{M_\odot}$$

Uncertainty in $M_{\mathrm{p,\,min}}$:

$$\Delta M_{\mathrm{p,\,min}} = M_{\mathrm{p,\,min}} \sqrt{\left(\frac{\Delta K}{K}\right)^2 + \left(\frac{2}{3}\frac{\Delta M_{\mathrm{s}}}{M_{\mathrm{s}}}\right)^2 + \left(\frac{1}{3}\frac{\Delta T}{T}\right)^2} = 0.091 \times 10^{-4}\,\mathrm{M_\odot}$$
$$M_{\mathrm{p,\,min}} = (6.93 \pm 0.09) \times 10^{-4}\,\mathrm{M_\odot}$$

(D01.2d) Using the value of $M_{\mathrm{p,\,min}}$ estimated in part (D01.2c), calculate the minimum value of the semi-major axis of the planet's orbit, a_{min}, in au and its uncertainty. **4**

Solution:

$$a_{\text{min}} = \left(\frac{T^2 G(M_s + M_p \sin i)}{4\pi^2}\right)^{1/3} = \left(\frac{T^2 G(M_s + M_{p,\,\text{min}})}{4\pi^2}\right)^{1/3}$$
$$= 0.0471\,\text{au}$$

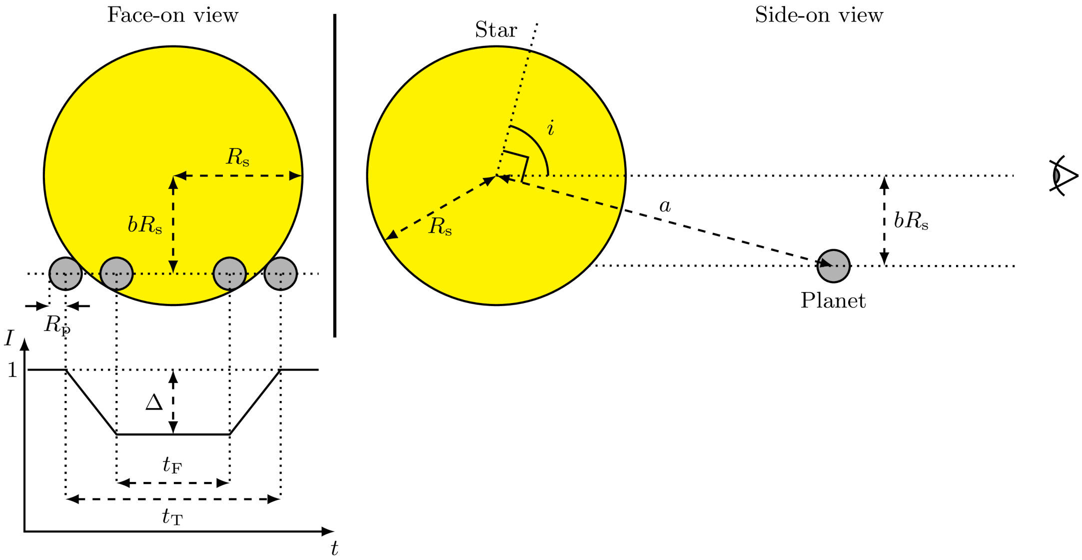
Uncertainty in a_{min}:

$$\Delta a_{\text{min}} = a_{\text{min}}\sqrt{\left(\frac{2}{3}\frac{\Delta T}{T}\right)^2 + \left(\frac{1}{3}\right)^2 \frac{(\Delta M_s)^2 + (\Delta M_{p,\,\text{min}})^2}{(M_s + M_{p,\,\text{min}})^2}}$$
$$= 0.0005\,\text{au}$$

$$a_{\text{min}} = (0.0471 \pm 0.0005)\,\text{au}$$

Transit method (without limb darkening)

The schematic diagram of a planet transit (not drawn to scale) is shown below. Initially, we shall assume the stellar disk to have a uniform average intensity with some intrinsic noise due to the star itself.



The lightcurve of the normalized intensity, I, as a function of time t is shown in the schematic diagram of the transit above. The average stellar intensity outside the transit is taken as unity. The maximum decrease in the intensity is given by Δ in the normalized light curve. For a uniformly bright stellar disk, the radius of the planet, R_p, is related to Δ as

$$\left(\frac{R_p}{R_s}\right)^2 = \Delta,$$

where R_s is the radius of the star.

The total duration of transit (when part or all of the planet covers the stellar disk) is given by t_T, while t_F gives the duration when the planet is fully in front of the stellar disk. The "impact parameter" b is the projected distance between the planet and centre of the stellar disk at the mid-point of the transit, in units of the stellar radius, R_s.

For a nearly edge-on star-planet orbit, the impact parameter is given by the formula

$$b = \left[\frac{(1-\sqrt{\Delta})^2 - (t_F/t_T)^2(1+\sqrt{\Delta})^2}{1-(t_F/t_T)^2}\right]^{1/2}$$

(D01.3) For the SP system, the stellar radius is known to be $R_\mathrm{s} = 1.20\,\mathrm{R}_\odot$, and the transit of the planet is indeed visible. Using the minimum orbital radius, a_min, estimated in part (D01.2d), find the minimum value, i_min, of the inclination angle. **3**

Solution:

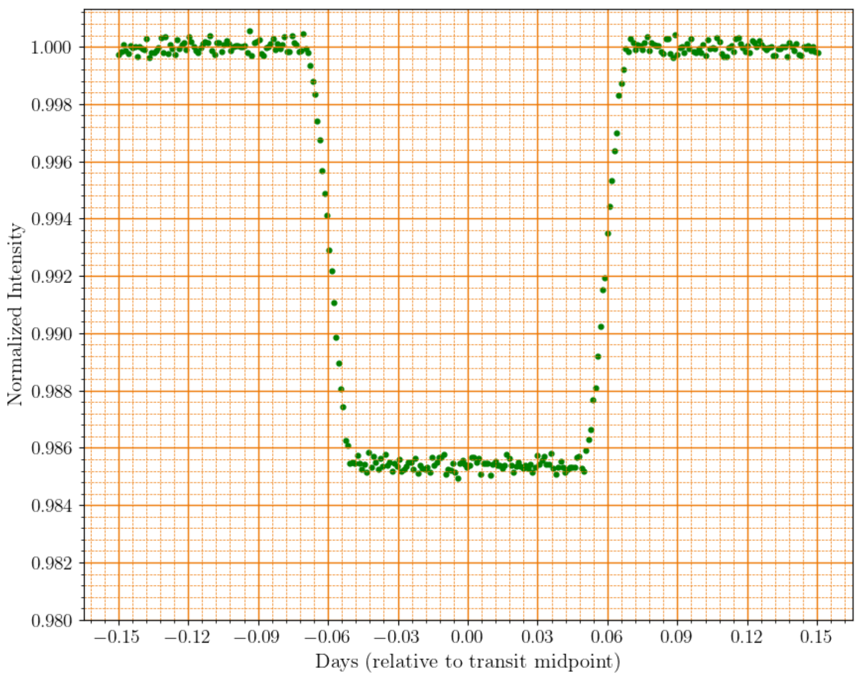
$$bR_\mathrm{s} = a\cos i \qquad \implies i = \cos^{-1}\frac{bR_\mathrm{s}}{a}$$

The minimum value of i corresponds to the maximum value of $b(=1)$ and the minimum value of a. Therefore,

$$i_\mathrm{min} = \cos^{-1}\left(\frac{b_\mathrm{max}R_\mathrm{s}}{a_\mathrm{min}}\right) = \cos^{-1}\left(\frac{R_\mathrm{s}}{a_\mathrm{min}}\right) = \cos^{-1}\left(\frac{1.20\,\mathrm{R}_\odot}{0.0471\,\mathrm{au}}\right)$$

$$i_\mathrm{min} = 83.20^\circ$$

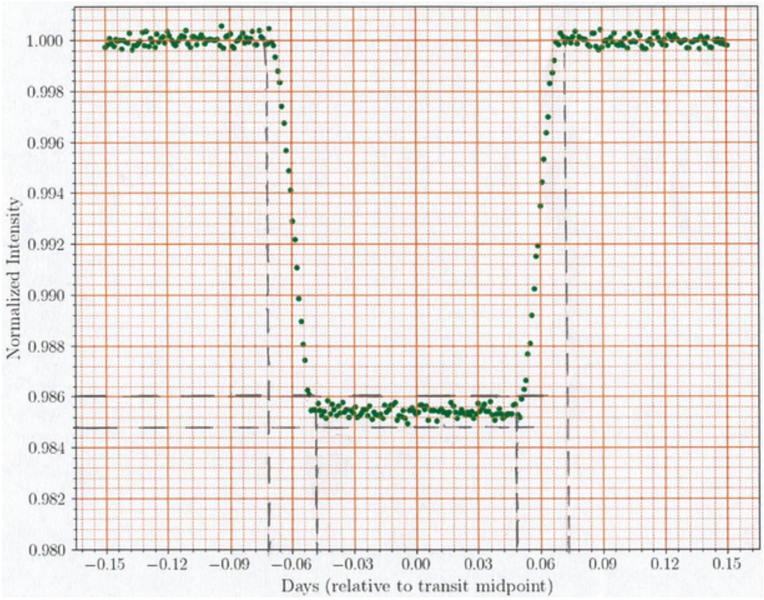
Assuming a stellar disk of uniform brightness, the transit lightcurve would look like as shown below.



(D01.4) Using the given lightcurve answer the following questions. For your reference the above lightcurve is also given in the Summary Answersheet.

(D01.4a) Estimate the values of t_T and t_F in days by marking appropriate readings on the graph. **3**

Solution:



In the figure of the transit curve, least count on time axis = 0.006 d, and on the intensity axis = 4×10^{-4}.
The measurements on the left and right edges of the transit for t_T are −0.072 d and 0.072 d.
The measurements on the left and right edges of the transit for t_F are −0.048 d and 0.048 d.
Therefore,

$$t_\mathrm{T} = 0.144\,\mathrm{d}$$
$$t_\mathrm{F} = 0.096\,\mathrm{d}$$

(D01.4b) Estimate the mean value of Δ, by marking appropriate readings on the graph and hence find R_p in units of $\mathrm{R}_\odot$. **2**

Solution:
The value of the dip is $(0.9860 + 0.9848)/2 = 0.9854$
Δ measured from the graph is $1 - 0.9854 = 0.0146$

$$\therefore \frac{R_\mathrm{p}}{R_\mathrm{s}} = \sqrt{\Delta} = 0.121$$
$$\implies R_\mathrm{p} = 0.121 R_\mathrm{s}$$
$$R_\mathrm{p} = 0.145\,\mathrm{R}_\odot$$

(D01.4c) Determine the value of i in degrees assuming the orbital radius to be a_min. **2**

Solution:
$$b = \left[\frac{(1-\sqrt{\Delta})^2 - (t_\mathrm{F}/t_\mathrm{T})^2(1+\sqrt{\Delta})^2}{1-(t_\mathrm{F}/t_\mathrm{T})^2}\right]^{1/2}$$
Plugging in values of Δ, t_T, and t_F, we get $b = 0.622$
$$bR_\mathrm{s} = a\cos i$$

Putting $a = a_{\text{min}}$,

$$i = \cos^{-1}\left(\frac{bR_s}{a_{\text{min}}}\right)$$

$$= \cos^{-1}\left(\frac{0.622 \times 1.2\,\mathrm{R}_\odot}{0.0471\,\mathrm{au}}\right)$$

$$i = 85.77^\circ$$

Introducing limb darkening

So far we have assumed the stellar disk to be uniformly bright. In reality, the observed brightness of the stellar disk is not uniform due to "limb darkening" — an optical effect where the central part of the stellar disk appears brighter than the edge, or the "limb".

The limb darkening effect can be measured by the relative intensity $J(\theta) \equiv \frac{I(\theta)}{I(0)}$, where θ is the angle between the normal to the stellar surface at a point and the line joining the observer to that point, $I(\theta)$ is the observed intensity of the stellar disk at that point ($I(0)$ being the intensity at the centre of the stellar disk). For a distant observer, θ varies from $\theta = 0$ (centre of the disk) to $\theta \approx 90^\circ$ (edge of the disk).

(D01.5) The table below gives measured $J(\theta)$ at a certain wavelength for the Sun. We shall assume that the same limb darkening profile holds for the star S.

θ	$J(\theta)$
0°	1.000
10°	0.994
15°	0.984

θ	$J(\theta)$
20°	0.971
25°	0.950
30°	0.943

θ	$J(\theta)$
40°	0.883
50°	0.794
60°	0.724

θ	$J(\theta)$
70°	0.595
80°	0.475
90°	0.312

The limb darkening profile can be modelled by a quadratic formula:

$$J(\theta) = 1 - a_1(1 - \cos\theta) - a_2(1 - \cos\theta)^2,$$

where a_1 and a_2 are two constants.

We shall estimate the unknown coefficients a_1 and a_2 from the given data by making a plot with suitable variables.

(D01.5a) Choose a pair of variables (x_1, y_1) which are suitable functions of θ and J, that you want to plot along x and y axes, respectively, to determine a_1 and a_2. Write the expressions for x_1 and y_1. **2**

If you need to define additional variables for additional plots, define them as (x_2, y_2), etc.

Solution:

Define $x_1 = 1 - \cos\theta$ and $y_1 = \frac{J - 1}{1 - \cos\theta} \equiv \frac{J - 1}{x_1}$

Then, $y_1 = -a_1 - a_2 x_1$

In this method a_1 and a_2 can be determined from the intercept and slope, respectively, of a best fit straight line to a plot of y_1 vs x_1.

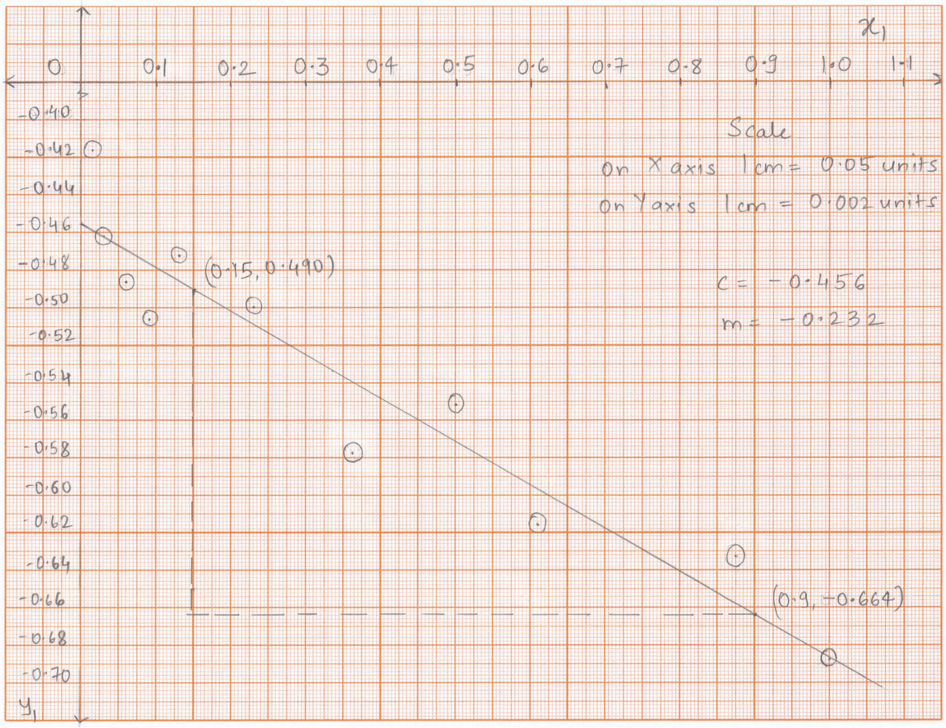
(D01.5b) Tabulate the values necessary for your plots. **4**

Solution:

θ	$J(\theta)$	x_1	y_1				
0°	1.000	0.000	×				
10°	0.994	0.015	−0.395				
15°	0.984	0.034	−0.470				
20°	0.971	0.060	−0.481				
20°	0.970	0.060	−0.497				
25°	0.950	0.094	−0.534				
30°	0.943	0.134	−0.425				
40°	0.883	0.234	−0.500				
40°	0.890	0.234	−0.470				
50°	0.794	0.357	−0.577				
60°	0.724	0.500	−0.552				
70°	0.595	0.658	−0.616				
80°	0.475	0.826	−0.635				
90°	0.312	1.000	−0.688				

(D01.5c) Plot the newly defined variables on the given graph paper (mark your graph as "D01.5c"). **7**

Solution:
Plot of y_1 vs x_1 with straight line fit:



(D01.5d) Obtain a_1 and a_2 from the plot. Uncertainties on the values are not needed. **7**

Solution:
Slope of best fit line = −0.232; Intercept = −0.456

$\implies a_1 = 0.46, \quad a_2 = 0.23$

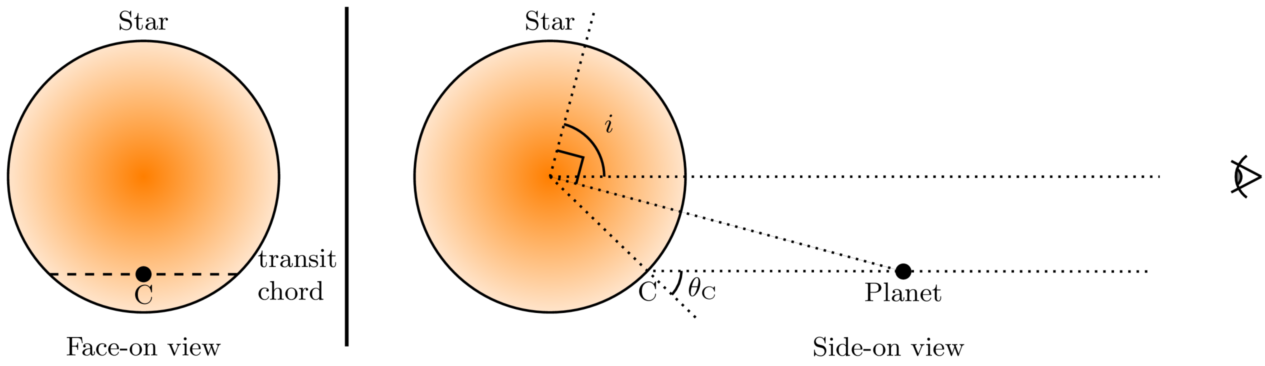
Transit in the presence of limb darkening

Now, we consider planetary transits across a limb darkened stellar disk. In the presence of limb darkening, which we shall model by the quadratic formula of $J(\theta)$ given above, the average observed intensity of the entire stellar disk (without any transit), $\langle I \rangle$, is given by:

$$\langle I \rangle = \left(1 - \frac{a_1}{3} - \frac{a_2}{6}\right) I(0)$$

Further, the dip in the light caused by the transiting planet now depends not only on the relative size of the planet and the star, $\left(\frac{R_p}{R_s}\right)$, but also on the intensity profile of the stellar disk along the transit chord, which in turn, depends on the angle of inclination, i.

The schematic diagram below (not drawn to scale) shows the configuration. Note that the brighter part of the star is shown in a darker shade, while the planet is shown as a black dot.



Here the relation between $\left(\frac{R_p}{R_s}\right)$ and the measured Δ from the light curve is

$$\Delta = \frac{I(\theta_C)}{\langle I \rangle} \left(\frac{R_p}{R_s}\right)^2,$$

where $I(\theta_C)$ is the intensity of the stellar disk at the midpoint of the transit chord (point C in the figure above), θ_C being the angle between the line of sight and the normal to the surface at that point. From the above it is obvious that for a given star, the same value of Δ can be produced by many combinations of the planet size, R_p, and the inclination angle i.

(D01.6) It is possible to uniquely determine both R_p and i by using data from transit lightcurves at two wavelengths, say, λ_B (blue) and λ_R (red). The limb darkening coefficients for these two wavelengths are given below:

Wavelength	a_1	a_2
λ_B	0.82	0.05
λ_R	0.24	0.20

(D01.6a) Choose the correct statement among the following that describes the relation between the maximum depth of the transit (Δ) for λ_B and the inclination angle (i) of the orbit and tick (✓) it in the Summary Answersheet. **2**

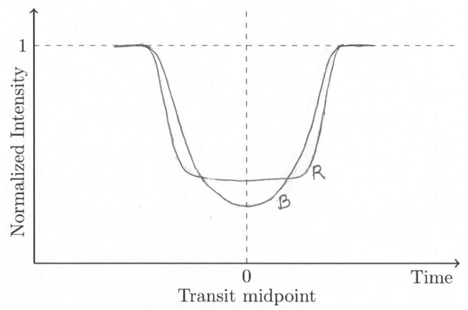
(A) Δ increases with decreasing i.
(B) Δ decreases with decreasing i.
(C) Δ is independent of i.

Solution:
As i decreases, the transit chord moves closer to the limb and the contribution of the midpoint of the transit to the dip in the average brightness

reduces. Therefore, Δ decreases with decreasing i.
Correct option: B

(D01.6b) The maximum depth of the transit (Δ) for the "SP system" was measured to be 0.0182 and 0.0159 for λ_B and λ_R, respectively. **4**
Draw schematic transit light curves for both λ_B and λ_R on the given grid and label the curves by "B" and "R" respectively. Assume that the total transit duration is same for both wavelengths. The curves need not be to scale, but should represent the shapes of the light curves correctly.

Solution:
Since the limb darkening effect is stronger for λ_B, it causes *lesser* reduction in the total light at the ingress and egress and more reduction at the transit centre, than for λ_R. Therefore, the curves must intersect.



(D01.7) We shall use a graphical method to find the values of R_p and i for the SP system using measurements of Δ at λ_B and λ_R.

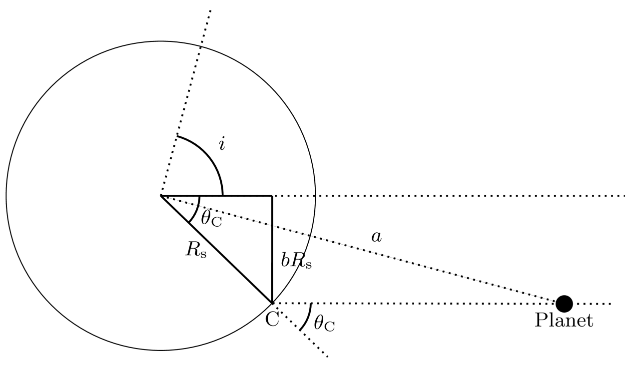
(D01.7a) Write an appropriate expression connecting the relevant variables that are to be plotted. (Hint: You may consider i or b, and R_p among the relevant variables.) **6**

Solution:

$$\Delta = \frac{I(\theta_C)}{\langle I \rangle}\left(\frac{R_p}{R_s}\right)^2$$

$$\implies \left(\frac{R_p}{R_s}\right)^2 = \frac{\Delta \langle I \rangle}{I(\theta_C)}$$

Here, $I(\theta_C)$ is a function of i. Further, $\langle I \rangle = \left(1 - \frac{a_1}{3} - \frac{a_2}{6}\right) I(0)$.



From the figure,

$$\sin\theta_C = \frac{bR_s}{R_s} = b = \frac{a\cos i}{R_s}$$

$$\implies \cos\theta_C = \sqrt{1-b^2} = \sqrt{1-\left(\frac{a\cos i}{R_s}\right)^2}$$

Therefore,

$$\begin{aligned} I(\theta_C) &= J(\theta_C)I(0) \\ &= \left[1 - a_1(1-\cos\theta_C) - a_2(1-\cos\theta_C)^2\right] I(0) \\ &= \left[1 - a_1\left(1-\sqrt{1-b^2}\right) - a_2\left(1-\sqrt{1-b^2}\right)^2\right] I(0) \end{aligned}$$

Thus,

$$\left(\frac{R_p}{R_s}\right) = \left[\frac{\Delta\langle I\rangle}{I(\theta_C)}\right]^{1/2}$$

$$\left(\frac{R_p}{R_s}\right) = \left[\frac{\Delta\left(1-\frac{a_1}{3}-\frac{a_2}{6}\right)}{1-a_1\left(1-\sqrt{1-b^2}\right)-a_2\left(1-\sqrt{1-b^2}\right)^2}\right]^{1/2}$$

$$\left(\frac{R_p}{R_s}\right) = \left[\frac{\Delta\left(1-\frac{a_1}{3}-\frac{a_2}{6}\right)}{1-a_1\left(1-\sqrt{1-\left(\frac{a\cos i}{R_s}\right)^2}\right)-a_2\left(1-\sqrt{1-\left(\frac{a\cos i}{R_s}\right)^2}\right)^2}\right]^{1/2}$$

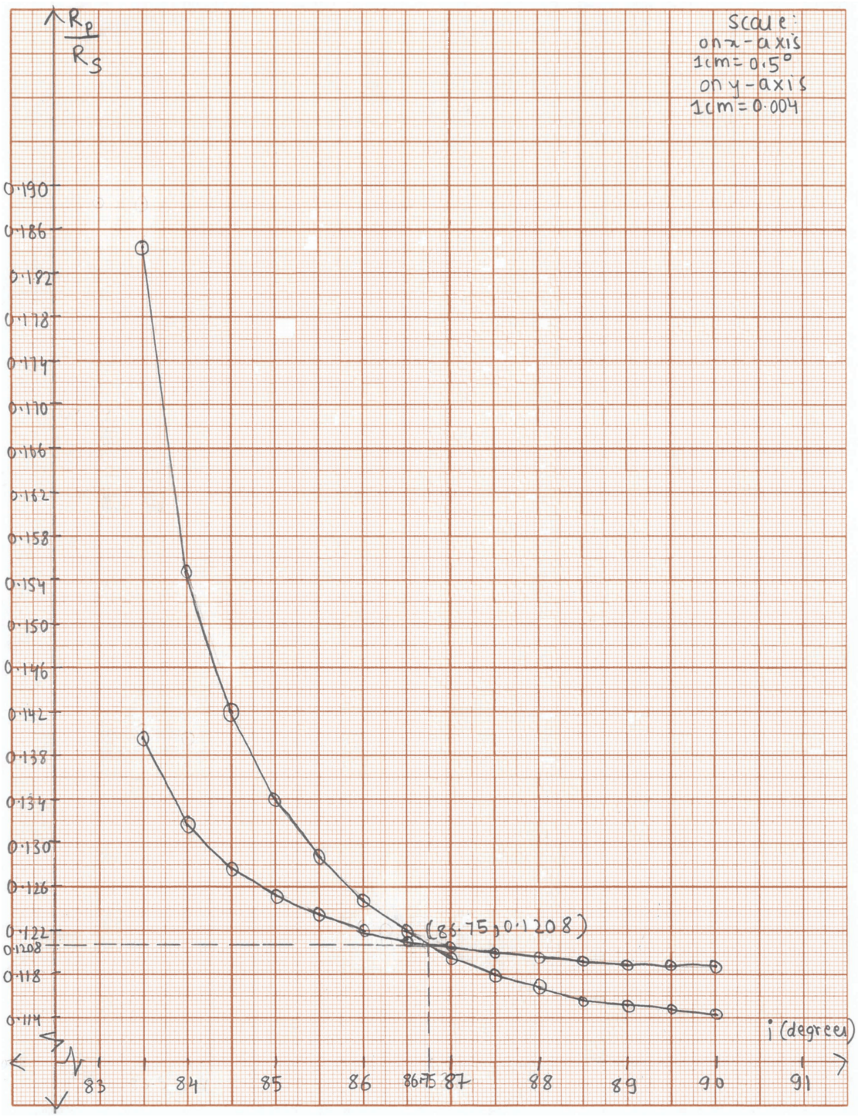
(D01.7b) Tabulate the appropriate quantities that are to be plotted. **5**

Solution:

i (°)	b	$(R_p/R_s)_{\lambda_B}$	$(R_p/R_s)_{\lambda_R}$
83.5	0.96	0.1842	0.1395
84.0	0.89	0.1547	0.1316
84.5	0.81	0.1419	0.1276
85.0	0.74	0.1341	0.1251
85.5	0.66	0.1287	0.1234
86.0	0.59	0.1248	0.1221
86.5	0.52	0.1218	0.1211
87.0	0.44	0.1196	0.1204
87.5	0.37	0.1178	0.1198
88.0	0.30	0.1165	0.1194
88.5	0.22	0.1155	0.1191
89.0	0.15	0.1149	0.1189
89.5	0.07	0.1145	0.1188
90.0	0.00	0.1143	0.1187

(D01.7c) Draw a suitable graph and mark it as "D01.7c". **7**

Solution:
One can plot the full range ($i_{min} \leq i \leq 90°$):



However, it is enough to plot around the point of intersection (can be deduced from the table above), to get a better resolution for the coordinates of the intersection point.

(D01.7d) Estimate the values of R_p (in $\mathrm{R}_\odot$) and i (in degrees) from the graph. **4**

Solution:
Intersection point of the graphs of $\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)$ vs i for λ_B and λ_R will give the values of both quantities.
Values deduced from either graph:

$$\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right) = 0.1208$$

$$\implies R_\mathrm{p} = 0.1208 \times 1.2\,\mathrm{R}_\odot$$

$$\boxed{R_\mathrm{p} = 0.145\,\mathrm{R}_\odot}$$

and

$i = 86.75°$

(D01.8) Based on the results obtained in this problem, indicate whether the planet P is "ROCKY" or "GASEOUS" by ticking (✓) the appropriate box in the Summary Answersheet. **2**

Solution:

$$\text{Estimated mass of the planet P } M_{\text{p}} = 6.93 \times 10^{-4}\,\text{M}_{\odot} = 1.389 \times 10^{27}\,\text{kg}$$

$$\text{Estimated radius of the planet P } R_{\text{p}} = 0.145\,\text{R}_{\odot} = 1.009 \times 10^{8}\,\text{m}$$

$$\therefore \text{Average density of the planet} = 320\,\text{kg}\,\text{m}^{-3}$$

The average density of the planet P is much less than that of Jupiter (a known gaseous planet). Hence the planet P is GASEOUS .

(D02) **Predicting arrival times of coronal mass ejections on Earth** **[60 marks]**

The Sun occasionally releases magnetized plasma, termed coronal mass ejections (CMEs), that originate from the surface of the Sun and propagate outwards. Accurate prediction of their arrival times at Earth is crucial for understanding and mitigating their potential effects on satellites orbiting the Earth. In this problem, we aim to predict the arrival times of CMEs by developing an empirical model, using the data of 10 CMEs. Throughout this problem, the distance between the Sun's surface and Earth is taken to be $214\,R_\odot$. Further, assume that the Sun is not rotating. Due to electromagnetic, gravitational and drag forces, CMEs experience a variable acceleration throughout their propagation. In the first two parts of this problem, we assume that the region between the Sun and the Earth is vacuum.

CMEs through vacuum

(D02.1) The initial velocity, u, at the solar surface ($= 1\,R_\odot$), the final velocity, v, upon reaching Earth, and the time to arrive at Earth after leaving the surface of the Sun (in hours), τ, are given for 10 CMEs in the following table.

CME Name	u (km s^{-1})	v (km s^{-1})	τ (h)
CME-A	804	470	74.5
CME-B	247	360	127.5
CME-C	523	396	103.5
CME-D	830	415	71.0
CME-E	665	400	104.5
CME-F	347	350	101.5
CME-G	446	375	99.5
CME-H	155	360	97.0
CME-I	1016	515	67.0
CME-J	683	410	54.0

(D02.1a) Calculate the average acceleration, a, for each CME in m s^{-2}. **3**

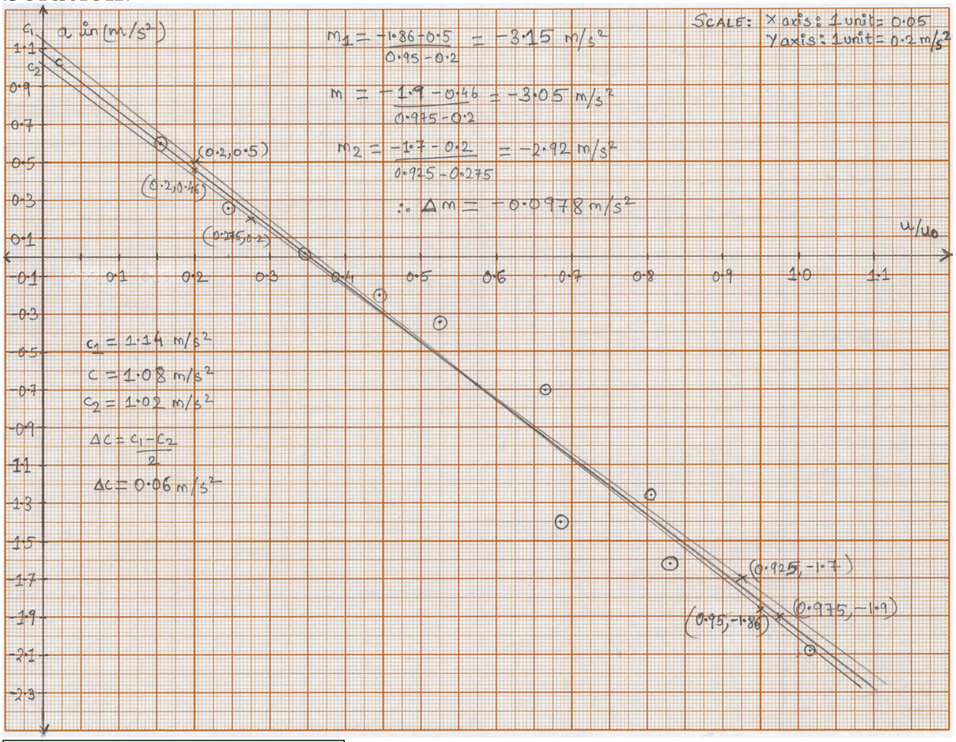
Solution:
Average acceleration (in m s^{-2}) is given by the following relation,

$$a = (v - u) \times 10^3/(\tau \times 3600)$$

CME Name	u (km s^{-1})	v (km s^{-1})	τ (h)	a (m s^{-2})
CME-A	804	470	74.5	−1.25
CME-B	247	360	127.5	0.246
CME-C	523	396	103.5	−0.341
CME-D	830	415	71.0	−1.62
CME-E	665	400	104.5	−0.704
CME-F	347	350	101.5	0.00821
CME-G	446	375	99.5	−0.198
CME-H	155	360	97.0	0.587
CME-I	1016	515	67.0	−2.08
CME-J	683	410	54.0	−1.40

(D02.1b) We assume an empirical model for the acceleration, a_{model}, of a CME, which **15** depends on its initial velocity, u, as $a_{\text{model}} = m\left(\frac{u}{u_0}\right) + \alpha$ where, a_{model} is expressed in $\mathrm{m\,s^{-2}}$, u is expressed in $\mathrm{km\,s^{-1}}$ and $u_0 = 1.00 \times 10^3\,\mathrm{km\,s^{-1}}$. Determine the constants m and α and their associated uncertainties using an appropriate graph (mark your graph as "D02.1b").

Solution:



$m = (-3.05 \pm 0.09)\,\mathrm{m\,s^{-2}}$

$\alpha = (1.08 \pm 0.06)\,\mathrm{m\,s^{-2}}$

Thus, the expression obtained by fitting a straight line to a vs u/u_0 is

$$a_{\text{model}} = -3.05\left(\frac{u}{u_0}\right) + 1.08$$

(D02.1c) For each CME, tabulate a_{model} in $\mathrm{m\,s^{-2}}$. Hence calculate the root-mean-square (rms) deviation of acceleration, δa_{rms}, between the calculated accelerations, a, and the model values, a_{model}. **4**

Solution:

CME Name	a_{model} ($\mathrm{m\,s^{-2}}$)
CME-A	−1.37
CME-B	0.327
CME-C	−0.515
CME-D	−1.45
CME-E	−0.948
CME-F	0.0217
CME-G	−0.280
CME-H	0.607
CME-I	−2.02
CME-J	−1.00

The root mean square (rms) deviation of acceleration is given by

$$\delta a_{\rm rms} = \sqrt{\frac{\sum_{i=1}^{10}(a_{{\rm model},i} - a_i)^2}{10}}$$

Using this expression we get, $\delta a_{\rm rms} = 0.176\,{\rm m\,s^{-2}}$

(D02.2) We consider two other CMEs: CME-1 and CME-2, with initial velocities, $u = 1044\,{\rm km\,s^{-1}}$ and $273\,{\rm km\,s^{-1}}$, respectively.

(D02.2a) Using the empirical model obtained in (D02.1b), calculate the predicted arrival times at Earth, $\tau_{1,\rm m}$ and $\tau_{2,\rm m}$ (in hours), for CME-1 and CME-2, respectively. **4**

Solution:
The time of arrival of a CME at the Earth, $\tau_{\rm m}$, can be calculated from the following relation.

$$s = u\tau_{\rm m} + \frac{1}{2}a_{\rm model}\tau_{\rm m}^2$$

However, this would yield two solutions for $\tau_{\rm m}$, only one of which will be for a positive final velocity (acceptable), and the other for a negative final velocity (not acceptable). Since acceleration is constant, the final velocity $v_{\rm m}$ at Earth at a distance $s = 214\,{\rm R}_\odot$ is given by

$$v_{\rm m} = \left|(u^2 + 2a_{\rm model}s)^{1/2}\right| = \left|\left[u^2 + 2\left(m\frac{u}{u_0} + \alpha\right)s\right]^{1/2}\right|$$

Using this velocity, we get the arrival time τ as

$$\tau_{\rm m} = \frac{v_{\rm m} - u}{a_{\rm model}} = \frac{v_{\rm m} - u}{m\frac{u}{u_0} + \alpha}$$

Using the given value of u, and the obtained values of m and α, we get

For CME-1:
$v_{1,\rm m} = 680.728\,{\rm km\,s^{-1}}$ which gives $\tau_{1,\rm m} = 48.0\,{\rm h}$

For CME-2:
$v_{1,\rm m} = 384.941\,{\rm km\,s^{-1}}$ which gives $\tau_{2,\rm m} = 126\,{\rm h}$

(D02.2b) The observed arrival times at Earth of CME-1 and CME-2 are 46.0 h and 74.5 h, respectively. The empirical model is considered to be VALID for a particular CME if its predicted arrival time is within 20% of its observed arrival time; otherwise, it is NOT VALID. Indicate the validity of the model for each CME by ticking (✓) the appropriate box in the Summary Answersheet. **2**

Solution:
The arrival time for CME-1 is $\tau_{1,\rm m} = 48.0\,{\rm h}$,
therefore the percentage error in the arrival time of CME-1 is,
$\Delta\tau_{1,\rm m} = \frac{48.0-46.0}{46.0} \times 100\% = 4.35\%$

Similarly, the arrival time for CME-2 is $\tau_{2,\rm m} = 126\,{\rm h}$,
hence the percentage error is arrival time for CME-2 is,
$\Delta\tau_{2,\rm m} = \frac{126-74.5}{74.5} \times 100\% = 69.1\%$

	VALID	NOT VALID
CME-1	✓	
CME-2		✓

CMEs in presence of solar wind

In reality, the space between the Sun and the Earth is permeated with the solar wind, which exerts a drag force on CMEs. This drag force can either decelerate or accelerate a CME, depending on the CME's velocity relative to that of the solar wind. To account for the solar wind's influence, we will use a "drag-only" model for distances $R_{\rm obs}(t) \geq R_0$, where R_0 is the distance beyond which the drag force becomes the dominant force affecting the CME's motion.

The distance from the surface of the Sun as determined from the "drag-only" model, $R_{\rm D}(t)$, and velocity, $V_{\rm D}(t)$, of a CME in this model is given by

$$R_{\rm D}(t) = \frac{S}{\gamma} \ln\left[1 + S\gamma(V_0 - V_{\rm s})(t - t_0)\right] + V_{\rm s}(t - t_0) + R_0$$

$$V_{\rm D}(t) = \frac{V_0 - V_{\rm s}}{1 + S\gamma(V_0 - V_{\rm s})(t - t_0)} + V_{\rm s}$$

where, $\gamma = 2 \times 10^{-8}\,{\rm km}^{-1}$, $V_{\rm s}$ is the constant speed of the solar wind, R_0 and V_0 are the distance and velocity, respectively, at time t_0, and S is the sign factor. $S = 1$ if $V_0 > V_{\rm s}$; $S = -1$ if $V_0 \leq V_{\rm s}$.

(D02.3) The tables below show the observed radial distance from the surface of the Sun, $R_{\rm obs}(t)$ (measured in $R_\odot$), as a function of time, t (in hours), for two CMEs: CME-3 and CME-4. The last data point in each table (D5 and P8, respectively) corresponds to the arrival time of the respective CME at Earth. For this part, assume $V_{\rm s} = 330\,{\rm km\,s^{-1}}$.

CME-3		
Data point	t (in h)	$R_{\rm obs}(t)$ (in $R_\odot$)
D1	0.200	6.36
D2	0.480	7.99
D3	1.22	11.99
D4	1.49	13.51
D5	58.05	214

CME-4		
Data point	t (in h)	$R_{\rm obs}(t)$ (in $R_\odot$)
P1	1.00	4.00
P2	3.00	6.00
P3	4.00	9.00
P4	5.00	11.0
P5	21.0	43.0
P6	50.0	100
P7	85.0	170
P8	111	214

We shall evaluate if the "drag-only" model satisfactorily predicts the arrival times of these CMEs. To use this model an appropriate choice of t_0, and corresponding R_0 and V_0 needs to be made.

(D02.3a) For CME-3, take the following two cases: **6**

(C1) t_0 is taken as the midpoint of the interval D1 – D2

(C2) t_0 is taken as the midpoint of the interval D3 – D4

Assume the velocity remains constant in each specific interval D1–D2 and D3–D4, but may differ between the two intervals.

Using t_0, R_0, and V_0, calculate the difference between the observed and the predicted radial distance, $\delta R_{\rm D} \equiv R_{\rm obs}(t) - R_{\rm D}(t)$ in units of $R_\odot$ at $t = 58.05\,{\rm h}$, for each of the two cases.

Solution:

Inserting, R_0, t_0, V_0, and V_s in the given expression

$$R_D(t) = \frac{S}{\gamma} \ln\left[1 + S\gamma(V_0 - V_s)(t - t_0)\right] + V_s(t - t_0) + R_0,$$

we get

Case C1:

$$R_0 = \frac{7.99 + 6.36}{2}$$

$$R_0 = 7.18\,\mathrm{R_\odot}$$

$$t_0 = \frac{0.20 + 0.48}{2} = 0.340\,\mathrm{h}$$

$$V_0 = \frac{(7.99 - 6.36) \times 695700}{(0.48 - 0.20) \times 3600}$$

$$V_0 = 1.12 \times 10^3\,\mathrm{km\,s^{-1}}$$

we get, $R_D = 210.61\,\mathrm{R_\odot}$ at $t = 58.05$ h.

Thus, $\boxed{\delta R_D = 214 - 210.61 = 3.39\,\mathrm{R_\odot}}$.

Case C2:

$$R_0 = \frac{11.99 + 13.51}{2} = 12.8\,\mathrm{R_\odot}$$

$$t_0 = \frac{1.22 + 1.49}{2} = 1.36\,\mathrm{h}$$

$$V_0 = \frac{(13.51 - 11.99) \times 695700}{(1.49 - 1.22) \times 3600}$$

$$V_0 = 1.09 \times 10^3\,\mathrm{km\,s^{-1}}$$

Using the expression above, $R_D = 210.86\,\mathrm{R_\odot}$ at t=58.05.

Thus, $\boxed{\delta R_D = 214 - 210.86 = 3.14\,\mathrm{R_\odot}}$.

(D02.3b) Evaluate $R_D(t)$ at points, P5, P6, P7, and P8 between the Sun and the Earth for CME-4 for the following two cases adopting the procedure similar to (D02.3a): **4**
(C3) t_0 is taken as the midpoint of the interval P1 – P2
(C4) t_0 is taken as the midpoint of the interval P3 – P4

Solution:

Case C3:

$$R_0 = \frac{6 + 4}{2} = 5\,\mathrm{R_\odot}$$

$$t_0 = \frac{3 + 1}{2} = 2\,\mathrm{h}$$

$$V_0 = \frac{(6 - 4) \times 695700}{(3 - 1) \times 3600} = 193\,\mathrm{km\,s^{-1}}$$

Case C4:

$$R_0 = \frac{11 + 9}{2} = 10\,\mathrm{R_\odot}$$

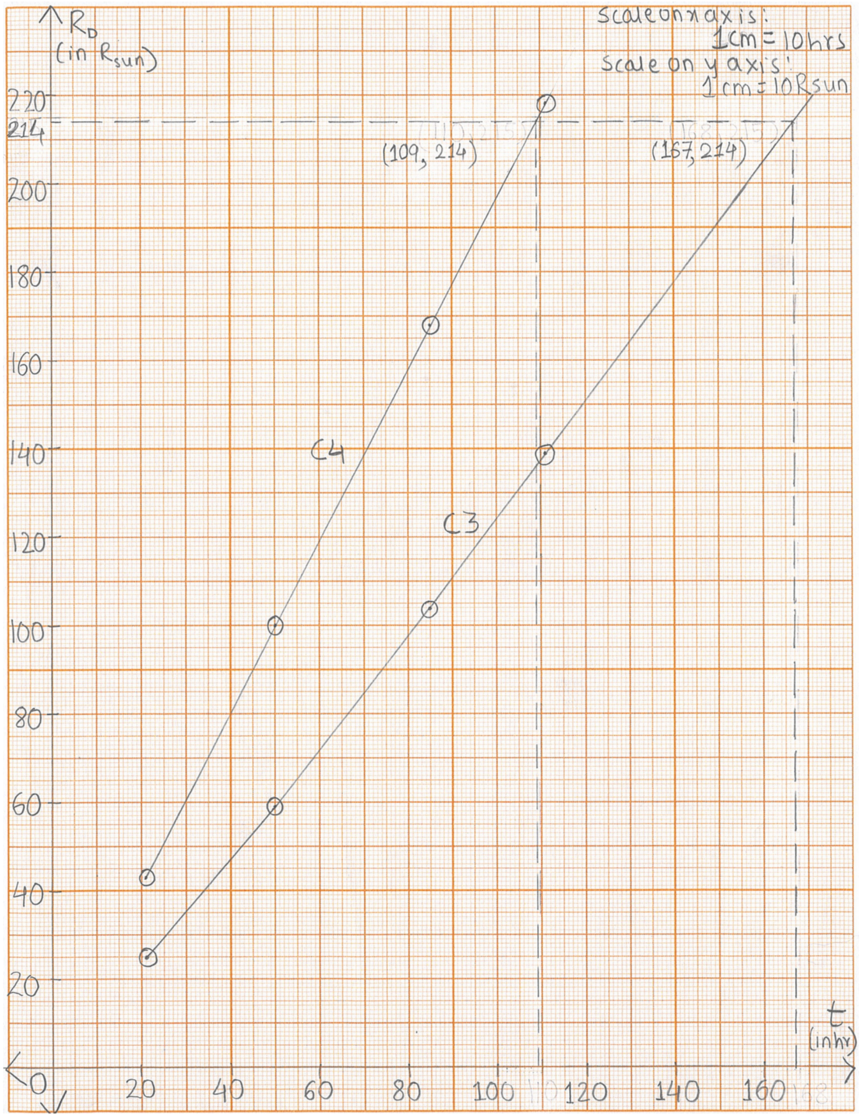
$$t_0 = \frac{5 + 4}{2} = 4.5\,\mathrm{h}$$

$$V_0 = \frac{(11 - 9) \times 695700}{(5 - 4) \times 3600} = 386\,\mathrm{km\,s^{-1}}$$

Data Point	t (in h)	C3	C4
		$R_\mathrm{D}(t)$ (in $\mathrm{R}_\odot$)	$R_\mathrm{D}(t)$ (in $\mathrm{R}_\odot$)
P5	21.0	25.1	42.8
P6	50.0	59.1	99.9
P7	85.0	104	168
P8	111	139	218

(D02.3c) Plot $R_\mathrm{D}(t)$ (in $\mathrm{R}_\odot$) vs t (in hours) for the two cases, C3 and C4, for CME-4 at points, P5, P6, P7, and P8 (mark your graph as "D02.3c"). On the same graph, draw smooth curves of $R_\mathrm{D}(t)$ for the above mentioned two cases. For this part, take the range of x axis from 0 to 180 h. **10**

Solution:
The plot below shows the data points of CME-4 and smooth variation of R_D extrapolated up to $220\,\mathrm{R}_\odot$.



(D02.3d) Using the graph, estimate the absolute difference, $|\delta\tau|$ between the actual arrival time of CME-4 at the Earth and its time of arrival predicted by the drag-only model, for each of the cases C3 and C4. **4**

Solution:
The arrival time of CME at Earth ($214\,R_\odot$) can be obtained by estimating the point of intersection between $R_D = 214\,R_\odot$ = constant and two smooth curves.
The points of intersection are 167 h and 109 h for the cases C3 and C4, respectively.
Thus, absolute time differences are:

C3: $|\delta\tau| = 56$ h

and

C4: $|\delta\tau| = 2$ h .

(D02.3e) Indicate whether the following statement is TRUE or FALSE by ticking (✓) the **1** appropriate box in the Summary Answersheet (no written justification needed): "The drag forces exerted by the solar wind on CMEs become dominant for CME-3 at an earlier time compared to CME-4."

Solution:
The statement is TRUE .
In the scenario of CME-3, the "drag-only" model predicts the arrival distance accurately when R_D is computed using the initial velocity, V_0 and initial distance, R_0 at 0.20 h and beyond.
While for CME-4, the "drag-only" model start accurately predicting the arrival times of CME-4 when R_D is calculated using V_0 and R_0 at 4 h after the launch of the CME.

(D02.4) Consider drag as the dominant force acting on 10 CMEs in part D02.1. Assume that the "drag-only" model is applicable from the surface of the Sun ($R_0 = 1\,R_\odot$) and beyond, for all CMEs. Estimate and tabulate the solar wind speed V_s in $\mathrm{km\,s^{-1}}$ for each CME. Further, estimate the average solar wind speed $V_{s,\,avg}$ for all 10 CMEs. 7

Solution:

$$V_D(t) = \frac{V_0 - V_s}{1 + S\gamma(V_0 - V_s)(t - t_0)} + V_s$$

$$v = \frac{u - V_s}{1 + S\gamma(u - V_s)\tau} + V_s.$$

Taking, $S = 1$ if $v < u$, else $S = -1$ The above expression is quadratic equation, thus, we get two values of V_s. If $v < u$, choose the solution with $V_s < u$. If $v > u$, choose the solution with $V_s > u$.
Calculating V_s using the above conditions for all ten CMEs.

CME Name	u ($\mathrm{km\,s^{-1}}$)	v ($\mathrm{km\,s^{-1}}$)	τ (h)	V_s ($\mathrm{km\,s^{-1}}$)
CME-A	804	470	74.5	337
CME-B	247	360	127.5	428
CME-C	523	396	103.5	314
CME-D	830	415	71.0	270
CME-E	665	400	104.5	303
CME-F	347	350	101.5	369
CME-G	446	375	99.5	305
CME-H	155	360	97.0	457
CME-I	1016	515	67.0	357
CME-J	683	410	54.0	248

Taking the average of the solar wind speed, V_s, for each case, we get
$V_{s,\,avg}$=339 $\mathrm{km\,s^{-1}}$